On the fundamental solution to the Kolmogorov equation & applications to financial market modeling

Francesca Anceschi (DIISM - Politecnica Marche)
f.anceschi@staff.univpm.it
joint project with A. Rebucci

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Kolmogorov-Fokker-Planck equation

$$\mathscr{K} u = \Delta_v u + v \cdot \nabla_x u - \partial_t u = 0, \qquad (v, x, t) \in \mathbb{R}^3$$

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- existence of the fundamental solution $\Gamma(v, x, t) = \frac{\sqrt{3}}{\pi t^2 \sigma^2} \exp\left(-\frac{2}{\sigma^2} \left(\frac{v^2}{t} + 3\frac{vx}{t^2} + 3\frac{x^2}{t^3} \right) \right)$, for $t > 0$;

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- hypoellipticity: let $\Omega \subset \mathbb{R}^{2d+1}$ and $\mathscr{K} u = f$. We say $\mathscr{K}$ is a hypoelliptic operator if $f \in C^\infty(\Omega)$, implies $u \in C^\infty(\Omega)$.

Importance of the fundamental solution

$$u(v,x,t) = \int_{\mathbb{R}^{2d}} \Gamma(v,x,t;\eta,\xi,\tau)\,\varphi(\eta,\xi)\; d\xi\, d\eta$$

- let $\varphi \in C_b(\mathbb{R}^{2d})$, then for every $(v_0, x_0) \in \mathbb{R}^{2d}$ there exists

$$\lim_{\substack{(v,x,t)\to(v_0,x_0,\tau)\\ t>\tau}} u(v,x,t) = \varphi(v_0,x_0);$$

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- let $\varphi \in C_b(\mathbb{R}^{2d})$, then there exists $T \in (\tau, T_1]$ such that u is a classical solution to the Cauchy problem

$$\begin{cases} \mathscr{K} u(v,x,t) = 0 & \text{in } \mathbb{R}^{2d} \times (\tau, T), \\ u(v,x,\tau) = \varphi(v,x) & \text{in } \mathbb{R}^{2d}. \end{cases}$$

Hörmander's operators

$$\mathscr{L}u(y,t) := \sum_{j=1}^{m} X_j^2 u(y,t) + X_0 u(y,t) \qquad (y,t) \in \mathbb{R}^{N+1}$$

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- Many other authors from then, among which we recall [Rothschild Stein 1976].

Kolmogorov operator in trace form

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  Hölder continuous coefficient in space and measurable in time $\Longrightarrow$ **a.e.** strong solution and Gaussian bounds

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- [Anceschi Rebucci 2021] measurable coefficients in space and time $\Longrightarrow$ existence of a **weak** fundamental solution and Gaussian upper and lower bounds

Kolmogorov operator in divergence form: setting

$$\mathscr{L}u(y,t) := div(A(y,t)u(y,t)) + Yu(y,t) = 0,$$
$$\forall (y,t) \in \mathbb{R}^{N+1},\ 1 \leq m_0 \leq N.$$

(A) A symmetric, real measurable entries and there exist λ, Λ s.t

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{m_0} a_{ij}(x,t)\xi_i\xi_j \leq \Lambda|\xi|^2 \qquad \forall (y,t) \in \mathbb{R}^{N+1},\ \xi \in \mathbb{R}^{m_0}$$

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(B) The *principal part operator* $\mathscr{K}$ is hypoelliptic and dilation invariant with respect to $(\delta_r)_{r>0}$, where $\mathscr{K}$ is

$$\mathscr{K} u(y,t) := \Delta_{m_0} u(y,t) + Yu(y,t).$$

Kolmogorov operator in divergence form: geometric setting

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- **Left translations:** for every (y, t), $(\xi, \tau) \in \mathbb{R}^{N+1}$ we define: $(y, t) \circ (\xi, \tau) = (\xi + E(\tau)y, t + \tau)$, where $E(s) = \exp(-sB)$;

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- **Family of dilations:** $\delta_r = \text{diag}(r\mathbb{I}_{m_0}, r^3\mathbb{I}_{m_1}, \ldots, r^{2\kappa+1}\mathbb{I}_{m_\kappa}, r^2)$;

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- **Family of slanted cylinders:** starting from the unit past cylinder $\mathcal{Q}_1 := B_1 \times B_1 \times \ldots \times B_1 \times (-1, 0)$ we define

$$\mathcal{Q}_r(z_0) := z_0 \circ (\delta_r(\mathcal{Q}_1)) = \{z \in \mathbb{R}^{N+1} : z = z_0 \circ \delta_r(\zeta), \zeta \in \mathcal{Q}_1\}$$

the cylinder centered at $z_0 \in \mathbb{R}^{N+1}$ and of radius r.

- **Hypoellipticity of** $\mathscr{K}$ is equivalent to assume:

$$B = \begin{pmatrix} \mathbb{O} & \mathbb{O} & \dots & \mathbb{O} & \mathbb{O} \\ B_1 & \mathbb{O} & \dots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & B_2 & \dots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & B_\kappa & \mathbb{O} \end{pmatrix} \qquad \begin{cases} m_0 \geq m_1 \geq \ldots \geq m_\kappa \geq 1 \\ \sum_{j=0}^{\kappa} m_j = N \end{cases}$$

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- $C(t) = C(t) = \int_0^t E(s)\, A_0\, E^T(s)\, ds > 0$ for every $t > 0$;

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- **Fundamental solution of $\mathscr{K}$**: $\Gamma_K(z,\zeta) = \Gamma_K(\zeta^{-1} \circ z, 0)$ for every $z = (y,t)$, $\zeta = (\xi,\tau) \in \mathbb{R}^{N+1}$, $z \neq \zeta$, where

$$\Gamma_K(z,0) = \begin{cases} \frac{(4\pi)^{-\frac{N}{2}}}{\sqrt{\det C(t)}} \exp\left(-\frac{1}{4}\langle C^{-1}(t)y, y\rangle - t\,\mathrm{tr}(B)\right), & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$

Kolmogorov operator in divergence form: functional setting

- [Albritton Armstrong Mourrat Novack] [Nyström Litsgard] Kolmogorov-Fokker-Planck equation
- [Anceschi Rebucci] Kolmogorov equation with κ commutators

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Let us consider $\Omega = \Omega_{m_0} \times \Omega_{N-m_0+1} \subset \mathbb{R}^{m_0} \times \mathbb{R}^{N-m_0+1}$, then $\mathcal{W}(\Omega)$ denotes the closure of $C_c^\infty(\overline{\Omega})$ in the norm $\| \cdot \|_{\mathcal{W}(\Omega)}$

$$\|u\|^2_{\mathcal{W}(\Omega)} = \|u\|^2_{L^2(\Omega_{N-m_0+1}; H^1_{x^{(0)}})} + \|Yu\|^2_{L^2(\Omega_{N-m_0+1}; H^{-1}_{x^{(0)}})},$$

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A function $u \in \mathcal{W}(\Omega)$ is a **weak solution** to $\mathscr{L} u = 0$ if for every $\varphi \in \mathcal{D}(\Omega), \varphi \geq 0$, we have

$$\int_\Omega -\langle ADu, D\varphi\rangle - uY\varphi + \langle b, Du\rangle\varphi + cu\varphi = 0.$$

The weak fundamental solution

A **weak fundamental solution** for $\mathscr{L}$ is a continuous positive function $\Gamma_L = \Gamma_L(x, t; \xi, \tau)$ defined for $t \in \mathbb{R}$, $0 \leq T_0 < \tau < t < T_1$ and any $x, \xi \in \mathbb{R}^N$ such that:

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[Lanconelli Pascucci Polidoro 2020] $\implies$ [Anceschi Rebucci 2022]

Gaussian bounds for the weak fundamental solution

Let $I = (T_0, T_1)$ be a bounded interval, then there exist four positive constants λ^+, λ^-, C^+, C^- such that

$$C^- \, \Gamma_K^{\lambda^-}(y,t;\xi,\tau) \le \Gamma_L(y,t;\xi,\tau) \le C^+ \, \Gamma_K^{\lambda^+}(y,t;\xi,\tau)$$

for every $(y,t),(\xi,\tau) \in \mathbb{R}^N \times (T_0, T_1)$ with $\tau < t$, where $\Gamma_K^{\lambda^*}$ denote the fundamental solution of $\mathscr{K}_{\lambda^*}$, where

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- Upper bounds: [Polidoro Pascucci 2003] [Lanconelli Pascucci 2017] [Lanconelli Pascucci Polidoro 2020]
- Lower bounds: [Lanconelli Pascucci Polidoro 2020]

Gaussian Upper bound

$$\Gamma(x,t;y,t_0) \leq \frac{c_1}{(t-t_0)^{\frac{Q}{2}}} \exp\left(-\frac{1}{c_1}\left|\delta^0_{(t-t_0)^{-\frac{1}{2}}}\left(y - e^{(t-t_0)B}x\right)\right|^2\right)$$

for any $0 < t - t_0 \leq 1$ and $x, y \in \mathbb{R}^N$.

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2. **Sobolev inequality:** the space $\mathcal{W}$
   - representation formula for the principal part operator $\mathscr{K}$;

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2. **Sobolev inequality:** the space $\mathcal{W}$
   - representation formula for the principal part operator $\mathscr{K}$;
   - potential estimates for convolutions of L^p functions with the fundamental solution of $\mathscr{K}$

Gaussian Lower bound

Then there exists $c_4 > 0$, only dependent on Q, λ, Λ such that

$$\Gamma(x, t; y, t_0) \geq \frac{c_4}{(t-t_0)^{\frac{Q}{2}}} e^{-c_4 \langle C^{-1}(t-t_0)\left(y-e^{(t-t_0)B}x\right), y-e^{(t-t_0)B}x\rangle}$$

for any $0 < t - t_0 \leq 1$ and $x, y \in \mathbb{R}^N$.

1. **Global Harnack inequality:** Harnack chains + control theory

Gaussian Lower bound

Then there exists $c_4 > 0$, only dependent on Q, λ, Λ such that

$$\Gamma(x,t;y,t_0) \geq \frac{c_4}{(t-t_0)^{\frac{Q}{2}}} e^{-c_4 \langle C^{-1}(t-t_0)\left(y-e^{(t-t_0)B}x\right), y-e^{(t-t_0)B}x\rangle}$$

for any $0 < t - t_0 \leq 1$ and $x, y \in \mathbb{R}^N$.

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2. **Intermediate lower bound:** Gaussian upp. bound + max. principle

Gaussian Lower bound

Let $\mathcal{Q}_1 := B_1 \times B_1 \times \ldots \times B_1 \times (-1, 0]$ and let u be a non-negative weak solution to $\mathscr{L} u = 0$ in $\Omega \supset \widetilde{\mathcal{Q}}_1$. Then

$$\sup_{\mathcal{Q}_-} u \leq C \inf_{\mathcal{Q}_+} u,$$

where $0 < \omega < 1$ and $0 < \rho < \frac{\omega}{\sqrt{2}}$. Finally, the constants C, ω, ρ only depend on $Q = m_0 + 3m_1 + \ldots + (2\kappa + 1)m_\kappa$ and on the ellipticity constants λ and Λ.

[Golse Imbert Mouhot Vasseur 2017] [Imbert Guerand 2021]
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4. Convergence on every compact subset $\implies$
   there exists a continuous Γ as $\varepsilon \to 0$.

Applications: Asian Options

Path-dependent options, whose pay-off depends on the average of the underlying stock price:

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- $Z_t = Z(S_t, A_t, t)$ price of the option, which depends also on

$$A_t = \int_0^t f(S_\tau)\, d\tau, \qquad t \in [0, T]$$

.

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- PDE approach, whose aim is to solve numerically the Cauchy problem associated to the no-arbitrage PDE [Barucci Polidoro Vespri 2001] [Vecer 2001] [Cibelli Polidoro Rossi 2019] [Anceschi Muzzioli Polidoro 2021]

Applications: Asian Options

$$\begin{cases} dS_t = \mu(S_t, A_t, t)S_t\,dt + \sigma(S_t, A_t, t)S_t\,dW_t \\ dB_t = r(S_t, A_t, t)B_t\,dt \\ dA_t = f(S_t)\,dt \end{cases}$$

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Replicating portfolio $Z_t = Z(S_t, A_t, t)$ **+ Itô's formula:**

$$\begin{cases} \frac{1}{2}\sigma^2 S^2 \partial_S^2 Z + f(S)\partial_A Z + r(S\partial_S Z - Z) + \partial_t Z = 0 \\ Z(S, A, T) = \varphi(S, A) \end{cases}$$

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$f(S) = \log(S)$ Geometric Average Asian Options

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$f(S) = S$ Arithmetic Average Asian Options

Thank you for your attention!