Self-similar solution for Hardy operator

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Plan: 0. Motivation and goals 1. F-K and O-U semigroups
2. Stationary distribution 3. Asymptotics' and applications

[5] *Self-similar solution for Hardy operator*, 2022,
KB, P. Kim, T. Jakubowski, D. Pilarczyk

[4] *Fractional Laplacian with Hardy potential*, 2019,
KB, T. Grzywny, T. Jakubowski, D. Pilarczyk

Motivation

For $d \geq 3$ and $\kappa \in \mathbb{R}$, consider the Cauchy problem for

$$\partial_t u(x,t) = \left(\Delta + \kappa|x|^{-2}\right) u(x,t), \quad x \in \mathbb{R}^d, \quad t > 0.$$

Baras and Goldstein [1]: no positive local-in-time solutions for $\kappa > (d-2)^2/4$ (instantaneous blow up). Vázquez and Zuazua [15]: for $0 < \kappa \leq (d-2)^2/4$ some solutions stabilize in L^2 toward

$$V(x,t) = t^{\delta - \frac{d}{2}} |x|^{-\delta} e^{-\frac{|x|^2}{4t}},$$

where $\delta = \frac{d-2}{2} - \sqrt{(d-2)^2/4 - \kappa}$. Pilarczyk [14] proved the stabilization in weighted L^q spaces. Note that V is *self-similar*:

$$V(x,t) := at^{(\delta-d)/2} V(t^{-1/2}x, 1).$$

Fractional Laplacian

Our goal: Consider $d \in \mathbb{N}$, $0 < \alpha < 2$ and

$$\partial_t u(x,t) = \left(\Delta^{\alpha/2} + \kappa |x|^{-\alpha}\right) u(x,t), \quad x \in \mathbb{R}^d, \quad t > 0,$$

where (excuse my notation)

$$\begin{aligned} \Delta^{\alpha/2} u(x) &:= -(-\Delta)^{\alpha/2} u(x) \\ &:= \lim_{\epsilon \to 0+} \int_{|y-x|>\epsilon} \left(u(y) - u(x)\right) \nu(x-y)\, dy, \end{aligned}$$

$\nu(z) = \mathcal{A}_{d,\alpha} |z|^{-d-\alpha}$, $z \in \mathbb{R}^d$ (a Lévy-measure density),

$$\mathcal{A}_{d,\alpha} = 2^\alpha \Gamma\big((d+\alpha)/2\big) \pi^{-d/2} / |\Gamma(-\alpha/2)|,$$

and, say, $u \in C_c^2(\mathbb{R}^d)$. For $u \in L^2(\mathbb{R}^d)$, let

$$\mathcal{E}[u] := \mathcal{E}(u,u) := \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))^2 \nu(x-y)\, dy\, dx.$$

Hardy identity (ground-state representation) in $L^2(\mathbb{R}^d)$

If $0 < \alpha < d$, $0 \le \delta \le d - \alpha$, $h(x) := |x|^{-\delta}$ and

$$\kappa_\delta = \frac{2^\alpha \Gamma\left(\frac{\delta+\alpha}{2}\right)\Gamma\left(\frac{d-\delta}{2}\right)}{\Gamma\left(\frac{\delta}{2}\right)\Gamma\left(\frac{d-\delta-\alpha}{2}\right)},$$

then for $u \in L^2(\mathbb{R}^d)$ we have

$$\begin{aligned}\mathcal{E}[u] =& \kappa_\delta \int_{\mathbb{R}^d} \frac{u(x)^2}{|x|^\alpha}\, dx \\ &+ \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left[\frac{u(x)}{h(x)} - \frac{u(y)}{h(y)}\right]^2 h(x)h(y)\nu(x-y)\, dy\, dx,\end{aligned}$$

see Bogdan, Dyda and Kim [3].

Or, see Frank, Lieb and Seiringer [7] for $u \in C_c^\infty(\mathbb{R}^d)$.

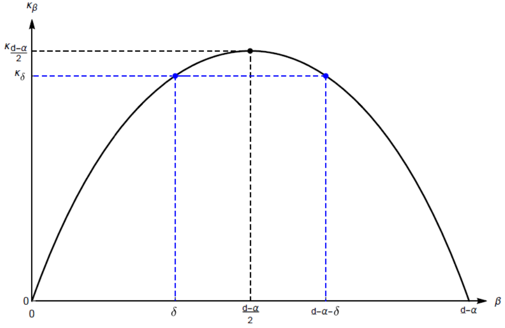
Hardy(-Rellich) inequality

The following fractional Hardy inequality is optimal in $L^2(\mathbb{R}^d)$:

$$\mathcal{E}[u] \geq \kappa_{(d-\alpha)/2} \int_{\mathbb{R}^d} \frac{u(x)^2}{|x|^\alpha}\, dx, \quad \text{where} \quad \kappa_{(d-\alpha)/2} = \frac{2^\alpha \Gamma\left(\frac{d+\alpha}{4}\right)^2}{\Gamma\left(\frac{d-\alpha}{4}\right)^2},$$

see Herbst [9], Beckner [2], or Yafaev [16].

Figure: The function $\beta \mapsto \kappa_\beta$.



Fix $\delta \in [0, (d-\alpha)/2]$ and the F-K semigroup $\tilde{P}_t$ generated by

$$\Delta^{\alpha/2} + \kappa_\delta |x|^{-\alpha}.$$

So, this is about the *subcritical and critical cases*.

The semigroup has density function $\tilde{p}(t,x,y)$, which is *self-similar*:

$$\tilde{p}(t,x,y) = t^{-d/\alpha} \tilde{p}\big(1, t^{-1/\alpha}x, t^{-1/\alpha}y\big), \qquad t > 0,\ x, y \in \mathbb{R}^d.$$

We would like to use $\tilde{p}_t(x,0)$ for our self-similar solution, but...

Also, if T is a (linear) isometry of $\mathbb{R}^d$, then

$$\tilde{p}(t, Tx, Ty) = \tilde{p}(t,x,y).$$

For $p(t,x,y) \sim \Delta^{\alpha/2}$ we have $p(t,x,y) \approx t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}$.

Recall that $0 \leq \delta \leq \frac{d-\alpha}{2}$ and $\tilde{p}(t,x,y) \sim \Delta^{\alpha/2} + \kappa_\delta |x|^{-\alpha}$.

Theorem (Bogdan, Grzywny, Jakubowski and Pilarczyk [4])

$$\tilde{p}(t,x,y) \approx \left(1 + t^{\delta/\alpha}|x|^{-\delta}\right)\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right)\left(1 + t^{\delta/\alpha}|y|^{-\delta}\right).$$

Also, $\tilde{p}(t,x,y)$ is continuous away from $x = 0$ and $y = 0$.

Recall $h(x) = |x|^{-\delta}$.

Theorem (Bogdan, Jakubowski, Kim, Pilarczyk [5])

$\Psi_t(x) := \lim_{y \to 0} \tilde{p}(t,x,y)/h(y)$ *exists,* $t > 0$, $x \in \mathbb{R}^d$.

In what follows we discuss related ideas and applications.

Auxiliary semigroups (1): Doob's conditioning

Theorem ([5])

The function $h(x) = |x|^{-\delta}$ is invariant for $\tilde{P}_t$:

$$\int_{\mathbb{R}^d} \tilde{p}(t,x,y)h(y)\,\mathrm{d}y = h(x), \qquad t > 0,\ x \in \mathbb{R}^d\,.$$

We define the Doob-conditioned (renormalized) kernel:

$$\rho_t(x,y) := \frac{\tilde{p}(t,x,y)}{h(x)h(y)}, \quad t > 0, \quad x,y \in \mathbb{R}^d_0 := \mathbb{R}^d \setminus \{0\}.$$

We are led to use the weight h^2 and we get:

$$\int_{\mathbb{R}^d} \rho_t(x,y)h^2(y)\,\mathrm{d}y = 1, \quad x \in \mathbb{R}^d_0, \quad t > 0,$$
$$\int_{\mathbb{R}^d} \rho_s(x,y)\rho_t(y,z)h^2(y)\,\mathrm{d}y = \rho_{t+s}(x,z), \quad x,z \in \mathbb{R}^d_0,\ s,t > 0.$$

We note the scaling, too: $\rho_{sr}(r^{1/\alpha}x, r^{1/\alpha}y) = r^{\frac{2\delta-d}{\alpha}}\rho_s(x,y)$.

Auxiliary semigroups (2): Ornstein-Uhlenbeck semigroup

We define a *transition density* with respect to $h^2(y)\,\mathrm{d}y$:

$$\ell_t(x,y) := \rho_{1-e^{-t}}(e^{-t/\alpha}x, y), \qquad t>0, \quad x,y\in\mathbb{R}^d_0.$$

Indeed, by Chapman-Kolmogorov and scaling of ρ,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \ell_s(x,y)\ell_t(y,z)h^2(y)\,\mathrm{d}y\\
&= \int_{\mathbb{R}^d} \rho_{1-e^{-s}}(e^{-s/\alpha}x,y)\rho_{1-e^{-t}}(e^{-t/\alpha}y,z)h^2(y)\,\mathrm{d}y\\
&= \int_{\mathbb{R}^d} \rho_{1-e^{-s}}(e^{-s/\alpha}x,y)(e^{-t})^{\frac{2\delta-d}{\alpha}}\rho_{e^t-1}(y,e^{t/\alpha}z)h^2(y)\,\mathrm{d}y\\
&= (e^{-t})^{\frac{2\delta-d}{\alpha}}\rho_{e^t-e^{-s}}(e^{-s/\alpha}x, e^{t/\alpha}z) = \rho_{1-e^{-s-t}}(e^{-(s+t)/\alpha}x,z).
\end{aligned}$$

For suitable functions f we define (O-U type semigroup):

$$L_t f(y) = \int_{\mathbb{R}^d} \ell_t(x,y)f(x)h^2(x)\,\mathrm{d}x.$$

Stationary density of L_t

If $\varphi \geq 0$ and $\int \varphi(x)h^2(x)\,\mathrm{d}x = 1$, then we say φ is a *density*.

The operators L_t preserve densities, so they are *Markov* in the setting of Komorowski [10], Lasota and Mackey [13], etc.

We say that density φ is *stationary* for L_t if $L_t\varphi = \varphi$.

Theorem

There is a unique stationary density φ for the operators L_t, $t > 0$.

Indeed, fix $t > 0$, $P = L_t$. For compactly supported densities f,

$$\begin{aligned} P^k f(y) &= \int_{\mathbb{R}^d} f(x)\rho_{1-e^{-kt}}(e^{-kt/\alpha}x, y)h^2(x)\,\mathrm{d}x \\ &\approx (1+|y|)^{-d-\alpha+\delta}, \qquad y \in \mathbb{R}^d_0, \quad k \in \mathbb{N}, \end{aligned}$$

since $\rho_t(x,y) \approx \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right)\left(t^{\delta/\alpha} + |x|^{\delta}\right)\left(t^{\delta/\alpha} + |y|^{\delta}\right).$

Stationary density of L_t, ctnd. (Existence)

Then we get (C1) in [10] for $B := \{x \in \mathbb{R}^d : 0 < |x| \le 1\}$:

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \int_B P^k f(y) h^2(y)\,\mathrm{d}y \approx \int_B (1+|y|)^{-d-\alpha+\delta} h^2(y)\,\mathrm{d}y > 0.$$

Also, (C2) in [10] is satisfied, since uniformly in k we have

$$\int_A P^k f(y) h^2(y)\,\mathrm{d}y \le C \int_A (1+|y|)^{-d-\alpha+\delta} h^2(y)\,\mathrm{d}y \to 0,$$

when $\int_A h^2(y)\,\mathrm{d}y \to 0$.

By [10, Theorem 3.1], a density φ exists satisfying $L_t\varphi = P\varphi = \varphi$. Or, we could use Schauder-Tychonoff fixed point theorem (and Dunford-Pettis), Krylov-Bogolioubov theorem [13], Doeblin or Dobrushin theorem [11, 8]. Worthwhile to look at Doob [6], too.

Stationary density of L_t, ctnd. (Uniqueness)

If ψ is a *different* stationary density, then $r := \varphi - \psi$ satisfies $r = Pr = Pr_+ - Pr_- \neq 0$ and we get this absurd:

$$\begin{aligned}\int_{\mathbb{R}^d} |r(x)| h^2(x)\,\mathrm{d}x &= \int_{\mathbb{R}^d} |Pr(x)| h^2(x)\,\mathrm{d}x \\ &< \int_{\mathbb{R}^d} P|r|(x) h^2(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} |r(x)| h^2(x)\,\mathrm{d}x,\end{aligned}$$

the last equation justified since P is Markov.

Since the operators L_t, $t > 0$, commute, they have the same stationary density, by the uniqueness: If $s > 0$ and $Q = L_s$, then $P(Q\varphi) = QP\varphi = (Q\varphi)$, and $Q\varphi$ is a density, so $Q\varphi = \varphi$.

Asymptotic stability

By the results of Kulik and Scheutzov [12, Theorem 1 and Remark 2], for every $x \in \mathbb{R}^d_0$ we get

$$\int_{\mathbb{R}^d} |\ell_t(x,y) - \varphi(y)|\, h^2(y)\,\mathrm{d}y \to 0 \text{ as } t \to \infty.$$

By cosmetics we get the following convergence in $L^1(h^2)$.

Lemma

We have $\int_{\mathbb{R}^d} |\rho_1(x,y) - \varphi(y)|\, h^2(y)\,\mathrm{d}y \to 0$ *as* $x \to 0$.

We bootstrap this to pointwise convergence and regularity of φ, as follows.

Regularization of the stationary density

Lemma

After modification on a set of Lebesgue measure zero, φ is continuous and radial on $\mathbb{R}^d$ with $\varphi(y) \approx (1+|y|)^{-d-\alpha+\delta}$, $y \in \mathbb{R}^d$.

Indeed, let's focus on the limit of $\varphi(y)$ as $y \to 0$:

$$\varphi(y) = L_t\varphi(y) = \int_{\mathbb{R}_0^d} \rho_{1-e^{-t}}\big(e^{-t/\alpha}x, y\big)\varphi(x)h^2(x)\,\mathrm{d}x$$

$$= \int_{\mathbb{R}_0^d} e^{t(d-2\delta)/\alpha}\rho_1\big(z, (1-e^{-t/\alpha})^{1/\alpha}y\big)\varphi\big((e^t-1)^{1/\alpha}z\big)h^2(z)\,\mathrm{d}z$$

$$\to \int_{\mathbb{R}_0^d} e^{t(d-2\delta)/\alpha}\varphi(z)\varphi\big((e^t-1)^{1/\alpha}z\big)h^2(z)\,\mathrm{d}z$$

$$= \int_{\mathbb{R}_0^d} \varphi((e^{-t})^{1/\alpha}x)\varphi\big((1-e^{-t})^{1/\alpha}x\big)h^2(x)\,\mathrm{d}x < \infty.$$

In particular, we may let, $\varphi(0) := \int_{\mathbb{R}^d} \varphi(2^{-1/\alpha}x)^2h^2(x)\,\mathrm{d}x$.

Continuous extension of $\rho_t(x, y)$

Theorem

$\rho_t(x, y)$ *has a continuous extension to* $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

Proof.

By scaling, Chapman-Kolmogorov, for $\mathbb{R}^d_0 \ni y, x \to 0$,

$$\begin{aligned}
\rho_1(x, y) &= 2^{\frac{d-2\delta}{\alpha}} \rho_2(2^{1/\alpha}x, 2^{1/\alpha}y) \\
&= 2^{\frac{d-2\delta}{\alpha}} \int \rho_1(2^{1/\alpha}x, z)\rho_1(z, 2^{1/\alpha}y)h^2(z)\,\mathrm{d}z \\
&\to 2^{\frac{d-2\delta}{\alpha}} \int \varphi(z)\rho_1(z, 2^{1/\alpha}y)h^2(z)\,\mathrm{d}z \\
&= \int \varphi(z)\rho_{1/2}(2^{-1/\alpha}z, y)h^2(z)\,\mathrm{d}z = L_{\ln 2}\varphi(y) = \varphi(y).
\end{aligned}$$

Thus, $\rho_1(0, y) := \lim_{x\to 0} \rho_1(x, y) = \varphi(y)$, for $y \neq 0$, etc. □

The self-similar solution $\Psi_t(x)$

By the theorem, for $t > 0$, $x \in \mathbb{R}^d$,

$$\Psi_t(x) := \lim_{y \to 0} \frac{\tilde{p}(t,x,y)}{h(y)} = \rho_t(0,x)h(x) = t^{\frac{2\delta - d}{\alpha}} \varphi\big(t^{-1/\alpha}x\big)\, h(x).$$

In particular,

$$\Psi_t(x) = t^{\frac{\delta - d}{\alpha}} \Psi_1(t^{-1/\alpha}x), \quad t > 0, \quad x \in \mathbb{R}^d.$$

Since $\rho_{t+s}(0,x) = \int_{\mathbb{R}^d} \rho_t(0,y)\rho_s(y,x)h^2(y)\,\mathrm{d}y$, we get

$$\int_{\mathbb{R}^d} \tilde{p}(s,y,x)\Psi_t(y)\,\mathrm{d}y = \Psi_{t+s}(x).$$

Further, $\int_{\mathbb{R}^d} \Psi_t(x)h(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \Psi_1(x)h(x)\,\mathrm{d}x = 1.$

Application to large time asymptotics

Recall $\Psi_t(x) = \lim_{y\to 0} \frac{\tilde{p}(t,x,y)}{h(y)}$. Let $H(x) = 1 \vee h(x) = 1 \vee |x|^{-\delta}$.

Our second main result is a large-time asymptotics for $\tilde{P}_t$.

Theorem

If $f \in L^1(H)$, $a := \int_{\mathbb{R}^d} f(x)h(x)\,\mathrm{d}x$, $u(t,x) := \tilde{P}_t f(x)$, $q \in [1,\infty)$,

$$\lim_{t\to\infty} t^{\frac{d-2\delta}{\alpha}(1-\frac{1}{q})}\|u(t,\cdot) - a\Psi_t\|_{q,h} = 0.$$

Here $\|f\|_{q,h} := \|f/h\|_{L^q(h^2)} = \left(\int_{\mathbb{R}^d} |f(x)|^q h^{2-q}(x)\,\mathrm{d}x\right)^{\frac{1}{q}}$.

This is about how fast the solutions $u(t,x) = \tilde{P}_t f(x)$ of

$$\begin{cases} \partial_t u(x,t) = \left(\Delta^{\alpha/2} + \kappa|x|^{-\alpha}\right) u(x,t), & x \in \mathbb{R}^d, \quad t > 0, \\ u(x,0) = f(x), \end{cases}$$

approach (a multiple of) the self-similar solution.

Potential of $\Psi_t(x)$

Corollary

For $0 \le \delta < \frac{d-\alpha}{2}$, $t > 0$, $x \in \mathbb{R}^d_0$, *we have*

$$\int_0^t \Psi_s(x)\,\mathrm{d}s = \frac{\Gamma(d/2)}{2\pi^{d/2}\kappa'_\delta}\left(|x|^{-(d-\delta-\alpha)} - \int_{\mathbb{R}^d} \tilde{p}(t,x,z)|z|^{-(d-\delta-\alpha)}\,\mathrm{d}z\right).$$

For $\delta = \frac{d-\alpha}{2}$, $t > 0$, $x \in \mathbb{R}^d_0$,

$$\int_0^t \Psi_s(x)\,\mathrm{d}s = \frac{\Gamma(d/2)}{\pi^{d/2}\kappa''_\delta}\left(|x|^{-\delta}\ln|x| - \int_{\mathbb{R}^d} \tilde{p}(t,z,x)|z|^{-\delta}\ln|z|\,\mathrm{d}z\right).$$

Here κ'_δ *and* κ''_δ *are the derivatives of* κ_δ.

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