Schauder estimates for Kolmogorov operators with coefficients measurable in time

INdAM Meeting "Kolmogorov Operators and their Applications"

Marco Bramanti

Politecnico di Milano

Cortona, 13-17 June 2022

- S. Biagi, M. Bramanti: Schauder estimates for Kolmogorov-Fokker-Planck operators with coefficients measurable in time and Hölder continuous in space. (*preprint*) https://arxiv.org/abs/2205.10270

- S. Biagi, M. Bramanti: Schauder estimates for Kolmogorov-Fokker-Planck operators with coefficients measurable in time and Hölder continuous in space. (*preprint*) https://arxiv.org/abs/2205.10270
- M. Bramanti, S. Polidoro: Fundamental solutions for Kolmogorov-Fokker-Planck operators with time-depending measurable coefficients. Mathematics in Engineering, 2020, 2(4): 734-771.

Introduction - Lanconelli Polidoro KFP operators

- Let us consider tha class of Kolmogorov-Fokker-Planck operators with linear drift studied by Lanconelli-Polidoro 1994:

$$\mathcal{L}u = \sum_{i,j=1}^{q} a_{ij}\partial^2_{x_i x_j} u + \sum_{k,j=1}^{N} b_{jk} x_k \partial_{x_j} u - \partial_t u, \qquad (x,t) \in \mathbb{R}^{N+1},$$

Introduction - Lanconelli Polidoro KFP operators

- Let us consider tha class of Kolmogorov-Fokker-Planck operators with linear drift studied by Lanconelli-Polidoro 1994:

$$\mathcal{L}u = \sum_{i,j=1}^{q} a_{ij} \partial^2_{x_i x_j} u + \sum_{k,j=1}^{N} b_{jk} x_k \partial_{x_j} u - \partial_t u, \qquad (x,t) \in \mathbb{R}^{N+1},$$

- The matrix $A_0 = (a_{ij})_{i,j=1}^{q}$ is symmetric and positive on $\mathbb{R}^q$, with $q < N$ ($\mathcal{L}$ is degenerate parabolic).

Introduction - Lanconelli Polidoro KFP operators

- Let us consider tha class of Kolmogorov-Fokker-Planck operators with linear drift studied by Lanconelli-Polidoro 1994:

$$\mathcal{L}u = \sum_{i,j=1}^{q} a_{ij}\partial^2_{x_i x_j} u + \sum_{k,j=1}^{N} b_{jk} x_k \partial_{x_j} u - \partial_t u, \qquad (x,t) \in \mathbb{R}^{N+1},$$

- The matrix $A_0 = (a_{ij})_{i,j=1}^{q}$ is symmetric and positive on $\mathbb{R}^q$, with $q < N$ ($\mathcal{L}$ is degenerate parabolic).
- The vector field:

$$Yu = \sum_{k,j=1}^{N} b_{jk} x_k \partial_{x_j} u - \partial_t u$$

is called the *drift*.

- The operator $\mathcal{L}$ is *hypoelliptic* if and only if the matrix $B = (b_{jk})_{j,k=1}^{N}$ has the following structure

$$B = \begin{pmatrix} * & * & \dots & * & * \\ B_1 & * & \dots & \dots & \dots \\ \mathbb{O} & B_2 & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & B_k & * \end{pmatrix}$$

where every block B_j is an $m_j \times m_{j-1}$ matrix of rank m_j, with

$$q = m_0 \geq m_1 \geq \ldots \geq m_k \geq 1 \quad \text{and} \quad m_0 + m_1 + \ldots + m_k = N,$$

while the blocks $*$ are arbitrary.

- The operator $\mathcal{L}$ is *hypoelliptic* if and only if the matrix $B = (b_{jk})_{j,k=1}^{N}$ has the following structure

$$B = \begin{pmatrix} * & * & \dots & * & * \\ B_1 & * & \dots & \dots & \dots \\ \mathbb{O} & B_2 & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & B_k & * \end{pmatrix}$$

where every block B_j is an $m_j \times m_{j-1}$ matrix of rank m_j, with

$$q = m_0 \geq m_1 \geq \ldots \geq m_k \geq 1 \quad \text{and} \quad m_0 + m_1 + \ldots + m_k = N,$$

while the blocks $*$ are arbitrary.

- In this case $\mathcal{L}$ is also *translation invariant* w.r.t. the Lie group operation in $\mathbb{R}^{N+1}$:

$$(y, s) \circ (x, t) = (x + E(t)y, t + s) \text{ with } E(t) = \exp(-tB).$$

- The operator $\mathcal{L}$ is *hypoelliptic* if and only if the matrix $B = (b_{jk})_{j,k=1}^{N}$ has the following structure

$$B = \begin{pmatrix} * & * & \dots & * & * \\ B_1 & * & \dots & \dots & \dots \\ \mathbb{O} & B_2 & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & B_k & * \end{pmatrix}$$

where every block B_j is an $m_j \times m_{j-1}$ matrix of rank m_j, with

$$q = m_0 \geq m_1 \geq \ldots \geq m_k \geq 1 \quad \text{and} \quad m_0 + m_1 + \ldots + m_k = N,$$

while the blocks $*$ are arbitrary.

- In this case $\mathcal{L}$ is also *translation invariant* w.r.t. the Lie group operation in $\mathbb{R}^{N+1}$:

$$(y, s) \circ (x, t) = (x + E(t)y, t + s) \text{ with } E(t) = \exp(-tB).$$

- We will call these operators *left invariant model operators*.

The family of homogeneous left invariant operators

- If, moreover, all the $*$ blocks in the matrix B are zero, then $\mathcal{L}$ is also 2-homogeneous w.r.t. a nondiagonal family of dilations:

$$\begin{aligned} D(\lambda)(x,t) &\equiv (D_0(\lambda)(x), \lambda^2 t) = (\lambda^{q_1} x_1, \dots, \lambda^{q_N} x_N, \lambda^2 t), \text{ with} \\ (q_1, \dots, q_N) &= (\underbrace{1, \dots, 1}_{m_0}, \underbrace{3, \dots, 3}_{m_1}, \dots, \underbrace{2k+1, \dots, 2k+1}_{m_k}). \end{aligned}$$

The family of homogeneous left invariant operators

- If, moreover, all the $*$ blocks in the matrix B are zero, then $\mathcal{L}$ is also 2-homogeneous w.r.t. a nondiagonal family of dilations:

$$\begin{aligned} D(\lambda)(x,t) &\equiv (D_0(\lambda)(x), \lambda^2 t) = (\lambda^{q_1} x_1, \dots, \lambda^{q_N} x_N, \lambda^2 t), \text{ with} \\ (q_1, \dots, q_N) &= (\underbrace{1, \dots, 1}_{m_0}, \underbrace{3, \dots, 3}_{m_1}, \dots, \underbrace{2k+1, \dots, 2k+1}_{m_k}). \end{aligned}$$

- The *homogeneous dimension* of $\mathbb{R}^N$ is the integer

$$Q = \sum_{i=1}^{N} q_i > N,$$

while $Q+2$ is the homogeneous dimension of $\mathbb{R}^{N+1}$.

The family of homogeneous left invariant operators

- If, moreover, all the $*$ blocks in the matrix B are zero, then $\mathcal{L}$ is also 2-homogeneous w.r.t. a nondiagonal family of dilations:

$$\begin{aligned} D(\lambda)(x,t) &\equiv (D_0(\lambda)(x), \lambda^2 t) = (\lambda^{q_1} x_1, \dots, \lambda^{q_N} x_N, \lambda^2 t), \text{ with} \\ (q_1, \dots, q_N) &= (\underbrace{1, \dots, 1}_{m_0}, \underbrace{3, \dots, 3}_{m_1}, \dots, \underbrace{2k+1, \dots, 2k+1}_{m_k}). \end{aligned}$$

- The *homogeneous dimension* of $\mathbb{R}^N$ is the integer

$$Q = \sum_{i=1}^{N} q_i > N,$$

while $Q+2$ is the homogeneous dimension of $\mathbb{R}^{N+1}$.

- We will call these operators *homogeneous model operators*.

Example (Kolmogorov operator, 1931)

Let

$$\mathcal{L}u = u_{x_1 x_1} + x_1 u_{x_2} - u_t \text{ for } x \in \mathbb{R}^2$$

Then

$$\begin{aligned} (y_1, y_2, s) \circ (x_1, x_2, t) &= (x_1 + y_1, x_2 + y_2 - ty_1, t + s) \\ D(\lambda)(x_1, x_2, t) &= \left(\lambda x_1, \lambda^3 x_2, \lambda^2 t\right). \\ Q &= 4 \text{ homogeneous dimension of } \mathbb{R}^2 \\ Q + 2 &= 6 \text{ homogeneous dimension of } \mathbb{R}^3. \end{aligned}$$

This is a homogeneous model operator.

KFP operators with variable coefficients

- Several Authors have studied KFP operators with *variable coefficients*:

$$\mathcal{L}u = \sum_{i,j=1}^{q} a_{ij}(x,t)\partial^2_{x_i x_j} u + \sum_{k,j=1}^{N} b_{jk} x_k \partial_{x_j} u - \partial_t u \quad \text{where:}$$

KFP operators with variable coefficients

- Several Authors have studied KFP operators with *variable coefficients*:

$$\mathcal{L}u = \sum_{i,j=1}^{q} a_{ij}(x,t)\partial^2_{x_i x_j} u + \sum_{k,j=1}^{N} b_{jk} x_k \partial_{x_j} u - \partial_t u \quad \text{where:}$$

- $A_0(x,t) = (a_{ij}(x,t))_{i,j=1}^{q}$ is symmetric uniformly positive on $\mathbb{R}^q$, with:

$$\nu |\xi|^2 \leq \sum_{i,j=1}^{q} a_{ij}(x,t)\xi_i \xi_j \leq \nu^{-1} |\xi|^2$$

for some $\nu > 0$, $\forall \xi \in \mathbb{R}^q$, $\forall x \in \mathbb{R}^N$, a.e. $t \in \mathbb{R}$;

KFP operators with variable coefficients

- Several Authors have studied KFP operators with *variable coefficients*:

$$\mathcal{L}u = \sum_{i,j=1}^{q} a_{ij}(x,t)\partial^2_{x_i x_j} u + \sum_{k,j=1}^{N} b_{jk} x_k \partial_{x_j} u - \partial_t u \quad \text{where:}$$

- $A_0(x,t) = (a_{ij}(x,t))_{i,j=1}^{q}$ is symmetric uniformly positive on $\mathbb{R}^q$, with:

$$\nu|\xi|^2 \le \sum_{i,j=1}^{q} a_{ij}(x,t)\xi_i\xi_j \le \nu^{-1}|\xi|^2$$

for some $\nu > 0$, $\forall \xi \in \mathbb{R}^q$, $\forall x \in \mathbb{R}^N$, a.e. $t \in \mathbb{R}$;
- B has the structure which makes the model operator with constant a_{ij} left invariant *or* left invariant and 2-homogeneous.

KFP operators with variable coefficients

- Several Authors have studied KFP operators with *variable coefficients*:

$$\mathcal{L}u = \sum_{i,j=1}^{q} a_{ij}(x,t)\partial^2_{x_i x_j} u + \sum_{k,j=1}^{N} b_{jk} x_k \partial_{x_j} u - \partial_t u \quad \text{where:}$$

- $A_0(x,t) = (a_{ij}(x,t))_{i,j=1}^{q}$ is symmetric uniformly positive on $\mathbb{R}^q$, with:

$$\nu |\xi|^2 \le \sum_{i,j=1}^{q} a_{ij}(x,t)\xi_i \xi_j \le \nu^{-1} |\xi|^2$$

  for some $\nu > 0$, $\forall \xi \in \mathbb{R}^q$, $\forall x \in \mathbb{R}^N$, a.e. $t \in \mathbb{R}$;
- B has the structure which makes the model operator with constant a_{ij} left invariant *or* left invariant and 2-homogeneous.
- We are interested in particular in *Schauder-type estimates*, that is a priori estimates on the Hölder norms of

$$u_{x_i x_j}, Yu, u_{x_i}, u \;\; (i,j = 1,\dots,q)$$

  in terms of the Hölder norm of $\mathcal{L}u$ and $\sup |u|$.

KFP operators with variable coefficients

- In 2006, Di Francesco-Polidoro proved Schauder-type estimates on bounded cylinders for operators with Hölder $a_{ij}(x,t)$ corresponding to left invariant model operators.

KFP operators with variable coefficients

- In 2006, Di Francesco-Polidoro proved Schauder-type estimates on bounded cylinders for operators with Hölder $a_{ij}(x,t)$ corresponding to left invariant model operators.
- Schauder-type estimates on $\mathbb{R}^N \times (0,T)$ for KFP operators have been proved by several authors, see Priola [2009], Imbert-Mouhot [2021].

KFP operators with variable coefficients

- In 2006, Di Francesco-Polidoro proved Schauder-type estimates on bounded cylinders for operators with Hölder $a_{ij}(x, t)$ corresponding to left invariant model operators.
- Schauder-type estimates on $\mathbb{R}^N \times (0, T)$ for KFP operators have been proved by several authors, see Priola [2009], Imbert-Mouhot [2021].
- Recent researches from the field of SDE suggest the importance of developing a theory allowing the coefficients a_{ij} to be *rough in t* (say, L^∞), and *Hölder continuous (in a suitable sense) only w.r.t. the space variables.*

KFP operators with variable coefficients

- In 2006, Di Francesco-Polidoro proved Schauder-type estimates on bounded cylinders for operators with Hölder $a_{ij}(x,t)$ corresponding to left invariant model operators.
- Schauder-type estimates on $\mathbb{R}^N \times (0,T)$ for KFP operators have been proved by several authors, see Priola [2009], Imbert-Mouhot [2021].
- Recent researches from the field of SDE suggest the importance of developing a theory allowing the coefficients a_{ij} to be *rough in t* (say, L^∞), and *Hölder continuous (in a suitable sense) only w.r.t. the space variables.*
- Hence one can try to prove Schauder-type estimates when the coefficients are Hölder continuous in x and only L^∞ in t.

Partial Schauder estimates for uniformly parabolic operators

- For *uniformly parabolic equations*,

$$Lu \equiv u_t - \sum_{i,j=1}^{N} a_{ij}(x,t)\, u_{x_i x_j},$$

the study of operators with $a_{ij}(x,t) \in L^\infty\left((0,T), C^\alpha\left(\mathbb{R}^N\right)\right)$ has an old history.

Partial Schauder estimates for uniformly parabolic operators

- For *uniformly parabolic equations*,

$$Lu \equiv u_t - \sum_{i,j=1}^{N} a_{ij}(x,t)\, u_{x_i x_j},$$

the study of operators with $a_{ij}(x,t) \in L^\infty\left((0,T), C^\alpha\left(\mathbb{R}^N\right)\right)$ has an old history.

- For $\Omega \subset \mathbb{R}^N$ and $\alpha \in (0,1)$, let

$$\begin{aligned} |f|_{C_x^\alpha(\Omega\times(0,T))} &= \operatorname*{supess}_{t\in(0,T)} \sup_{x_1,x_2\in\Omega, x_1\neq x_2} \frac{|f(x_1,t) - f(x_2,t)|}{|x_1 - x_2|^\alpha}; \\ \|f\|_{C_x^\alpha(\Omega\times(0,T))} &= |f|_{C^\alpha(\Omega\times(0,T))} + \|f\|_{L^\infty(\Omega\times(0,T))}. \end{aligned}$$

Partial Schauder estimates for uniformly parabolic operators

- For *uniformly parabolic equations*,

$$Lu \equiv u_t - \sum_{i,j=1}^{N} a_{ij}(x,t)\, u_{x_i x_j},$$

  the study of operators with $a_{ij}(x,t) \in L^\infty\left((0,T), C^\alpha\left(\mathbb{R}^N\right)\right)$ has an old history.
- For $\Omega \subset \mathbb{R}^N$ and $\alpha \in (0,1)$, let

$$\begin{aligned} |f|_{C_x^\alpha(\Omega\times(0,T))} &= \operatorname*{supess}_{t\in(0,T)} \sup_{x_1,x_2\in\Omega, x_1\neq x_2} \frac{|f(x_1,t)-f(x_2,t)|}{|x_1-x_2|^\alpha}; \\ \|f\|_{C_x^\alpha(\Omega\times(0,T))} &= |f|_{C^\alpha(\Omega\times(0,T))} + \|f\|_{L^\infty(\Omega\times(0,T))}. \end{aligned}$$

- In 1969, Brandt proved *interior partial Schauder estimates*: for $\|a_{ij}\|_{C_x^\alpha(\Omega\times(0,T))} < \infty$ and $\Omega' \Subset \Omega$,

$$|u_{x_i x_j}|_{C_x^\alpha(\Omega'\times(0,T))} \leq c\left\{|Lu|_{C_x^\alpha(\Omega\times(0,T))} + \|u\|_{C^0(\Omega\times(0,T))}\right\}.$$

Partial Schauder estimates for uniformly parabolic operators

- In 1980, Knerr proved that, under the same assumptions,

$$\sup_{\substack{(x_1,t_1),(x_2,t_2)\\ \in\Omega'\times(0,T)}} \frac{\left|u_{x_i x_j}(x_1,t_1)-u_{x_i x_j}(x_2,t_2)\right|}{\left|x_1-x_2\right|^{\alpha}+\left|t_1-t_2\right|^{\alpha/2}}$$

$$\leq c\left\{\left|Lu\right|_{C_x^{\alpha}(\Omega\times(0,T))}+\left\|u\right\|_{C^0(\Omega\times(0,T))}\right\}.$$

Partial Schauder estimates for uniformly parabolic operators

- In 1980, Knerr proved that, under the same assumptions,

$$\sup_{\substack{(x_1,t_1),(x_2,t_2)\\ \in\Omega'\times(0,T)}} \frac{\left|u_{x_ix_j}(x_1,t_1)-u_{x_ix_j}(x_2,t_2)\right|}{\left|x_1-x_2\right|^{\alpha}+\left|t_1-t_2\right|^{\alpha/2}}$$

$$\leq c\left\{\left|Lu\right|_{C_x^{\alpha}(\Omega\times(0,T))}+\left\|u\right\|_{C^0(\Omega\times(0,T))}\right\}.$$

- Lieberman, 1992, gives a unified presentation of these results.

Partial Schauder estimates for uniformly parabolic operators

- In 1980, Knerr proved that, under the same assumptions,

$$\sup_{\substack{(x_1,t_1),(x_2,t_2)\\ \in\Omega'\times(0,T)}} \frac{\left|u_{x_ix_j}(x_1,t_1)-u_{x_ix_j}(x_2,t_2)\right|}{|x_1-x_2|^{\alpha}+|t_1-t_2|^{\alpha/2}}$$

$$\leq c\left\{|Lu|_{C_x^{\alpha}(\Omega\times(0,T))}+\|u\|_{C^0(\Omega\times(0,T))}\right\}.$$

- Lieberman, 1992, gives a unified presentation of these results.
- Krylov and Priola, 2010, have extended partial Schauder estimates (on the whole space) to parabolic operators with lower order unbounded terms.

Partial Schauder estimates for uniformly parabolic operators

- In 1980, Knerr proved that, under the same assumptions,

$$\sup_{\substack{(x_1,t_1),(x_2,t_2)\\ \in\Omega'\times(0,T)}} \frac{\left|u_{x_ix_j}(x_1,t_1)-u_{x_ix_j}(x_2,t_2)\right|}{\left|x_1-x_2\right|^{\alpha}+\left|t_1-t_2\right|^{\alpha/2}}$$

$$\leq c\left\{\left|Lu\right|_{C_x^{\alpha}(\Omega\times(0,T))}+\left\|u\right\|_{C^0(\Omega\times(0,T))}\right\}.$$

- Lieberman, 1992, gives a unified presentation of these results.
- Krylov and Priola, 2010, have extended partial Schauder estimates (on the whole space) to parabolic operators with lower order unbounded terms.
- Lorenzi, 2011, has proved similar global estimates for operators with possibly unbounded coefficients.

Partial Schauder estimates for uniformly parabolic operators

- In 1980, Knerr proved that, under the same assumptions,

$$\sup_{\substack{(x_1,t_1),(x_2,t_2) \\ \in\Omega'\times(0,T)}} \frac{\left|u_{x_i x_j}(x_1,t_1)-u_{x_i x_j}(x_2,t_2)\right|}{\left|x_1-x_2\right|^{\alpha}+\left|t_1-t_2\right|^{\alpha/2}}$$

$$\leq \quad c\left\{\left|Lu\right|_{C_x^{\alpha}(\Omega\times(0,T))}+\left\|u\right\|_{C^0(\Omega\times(0,T))}\right\}.$$

- Lieberman, 1992, gives a unified presentation of these results.
- Krylov and Priola, 2010, have extended partial Schauder estimates (on the whole space) to parabolic operators with lower order unbounded terms.
- Lorenzi, 2011, has proved similar global estimates for operators with possibly unbounded coefficients.
- Dong-Kim, 2019, has proved partial Schauder estimates for operators with coefficients merely measurable w.r.t. *several* variables.

Partial Schauder estimates for degenerate KFP operators

- We want to establish partial Schauder estimates for degenerate KFP operators on $S_T = \mathbb{R}^N \times (-\infty, T)$.

Partial Schauder estimates for degenerate KFP operators

- We want to establish partial Schauder estimates for degenerate KFP operators on $S_T = \mathbb{R}^N \times (-\infty, T)$.
- We consider KFP operators

$$\mathcal{L}u = \sum_{i,j=1}^{q} a_{ij}(x,t)\partial^2_{x_i x_j} u + \sum_{k,j=1}^{N} b_{jk} x_k \partial_{x_j} u - \partial_t u \quad \text{where:}$$

Partial Schauder estimates for degenerate KFP operators

- We want to establish partial Schauder estimates for degenerate KFP operators on $S_T = \mathbb{R}^N \times (-\infty, T)$.
- We consider KFP operators

$$\mathcal{L}u = \sum_{i,j=1}^{q} a_{ij}(x,t)\partial^2_{x_i x_j} u + \sum_{k,j=1}^{N} b_{jk} x_k \partial_{x_j} u - \partial_t u \quad \text{where:}$$

- (H1) $A_0(x,t) = (a_{ij}(x,t))_{i,j=1}^{q}$ is symmetric uniformly positive on $\mathbb{R}^q$, with:

$$\nu|\xi|^2 \leq \sum_{i,j=1}^{q} a_{ij}(x,t)\xi_i\xi_j \leq \nu^{-1}|\xi|^2$$

for some $\nu > 0$, $\forall \xi \in \mathbb{R}^q$, $\forall x \in \mathbb{R}^N$, a.e. $t \in \mathbb{R}$;

Partial Schauder estimates for degenerate KFP operators

- We want to establish partial Schauder estimates for degenerate KFP operators on $S_T = \mathbb{R}^N \times (-\infty, T)$.
- We consider KFP operators

$$\mathcal{L}u = \sum_{i,j=1}^{q} a_{ij}(x,t)\partial^2_{x_i x_j} u + \sum_{k,j=1}^{N} b_{jk} x_k \partial_{x_j} u - \partial_t u \quad \text{where:}$$

- (H1) $A_0(x,t) = (a_{ij}(x,t))_{i,j=1}^{q}$ is symmetric uniformly positive on $\mathbb{R}^q$, with:

$$\nu|\xi|^2 \le \sum_{i,j=1}^{q} a_{ij}(x,t)\xi_i\xi_j \le \nu^{-1}|\xi|^2$$

  for some $\nu > 0$, $\forall \xi \in \mathbb{R}^q$, $\forall x \in \mathbb{R}^N$, a.e. $t \in \mathbb{R}$;
- (H2) B has the structure which makes the model operator which constant a_{ij} *left invariant and* 2-*homogeneous*.

Partial Schauder estimates for degenerate KFP operators

- Under our assumptions, the function

$$\rho(\xi) = \rho(x,t) := \|x\| + \sqrt{|t|} = \sum_{i=1}^{N} |x_i|^{1/q_i} + \sqrt{|t|}$$

is a *homogeneous norm* in $\mathbb{R}^{N+1}$, and

$$d(\xi,\eta) := \rho(\eta^{-1} \circ \xi)$$

is a left-invariant, 1-homogeneous *quasi-distance* on $\mathbb{R}^{N+1}$.

Partial Schauder estimates for degenerate KFP operators

- Under our assumptions, the function

$$\rho(\xi) = \rho(x,t) := \|x\| + \sqrt{|t|} = \sum_{i=1}^{N} |x_i|^{1/q_i} + \sqrt{|t|}$$

is a *homogeneous norm* in $\mathbb{R}^{N+1}$, and

$$d(\xi,\eta) := \rho(\eta^{-1} \circ \xi)$$

is a left-invariant, 1-homogeneous *quasi-distance* on $\mathbb{R}^{N+1}$.

- Note that, for $\xi = (x,t)\,, \eta = (y,s)\,,$

$$d(\xi,\eta) := \rho(\eta^{-1} \circ \xi) = \|x - E(t-s)y\| + \sqrt{|t-s|}$$

with $E(t) = \exp(-tB)$, but for $s = t$ one simply has

$$d((x,t)\,,(y,t)) = \|x-y\|\,.$$

Partial Schauder estimates for degenerate KFP operators

- For $S_T \equiv \mathbb{R}^N \times (-\infty, T)$, we will use the norms:

$$
\begin{aligned}
|f|_{C^\alpha(S_T)} &= \sup_{\xi_1,\xi_2 \in S_T, \xi_1 \neq \xi_2} \frac{|f(\xi_1) - f(\xi_2)|}{d(\xi_1,\xi_2)^\alpha} \\
|f|_{C^\alpha_x(S_T)} &= \operatorname{supess}_{t<T} \sup_{x_1,x_2 \in \mathbb{R}^N, x_1 \neq x_2} \frac{|f(x_1,t) - f(x_2,t)|}{\|x_1 - x_2\|^\alpha}. \\
\|f\|_{C^\alpha_x(S_T)} &= |f|_{C^\alpha_x(S_T)} + \|f\|_{L^\infty(S_T)}; \\
\|f\|_{C^\alpha(S_T)} &= |f|_{C^\alpha(S_T)} + \|f\|_{C^0(S_T)}
\end{aligned}
$$

Partial Schauder estimates for degenerate KFP operators

- For $S_T \equiv \mathbb{R}^N \times (-\infty, T)$, we will use the norms:

$$
\begin{aligned}
|f|_{C^\alpha(S_T)} &= \sup_{\xi_1,\xi_2\in S_T,\xi_1\neq\xi_2} \frac{|f(\xi_1)-f(\xi_2)|}{d(\xi_1,\xi_2)^\alpha} \\
|f|_{C^\alpha_x(S_T)} &= \operatorname{supess}_{t<T} \sup_{x_1,x_2\in\mathbb{R}^N,x_1\neq x_2} \frac{|f(x_1,t)-f(x_2,t)|}{\|x_1-x_2\|^\alpha}. \\
\|f\|_{C^\alpha_x(S_T)} &= |f|_{C^\alpha_x(S_T)} + \|f\|_{L^\infty(S_T)}; \\
\|f\|_{C^\alpha(S_T)} &= |f|_{C^\alpha(S_T)} + \|f\|_{C^0(S_T)}
\end{aligned}
$$

- Our last assumption is:

Partial Schauder estimates for degenerate KFP operators

- For $S_T \equiv \mathbb{R}^N \times (-\infty, T)$, we will use the norms:

$$
\begin{aligned}
|f|_{C^\alpha(S_T)} &= \sup_{\xi_1,\xi_2 \in S_T, \xi_1 \neq \xi_2} \frac{|f(\xi_1) - f(\xi_2)|}{d(\xi_1,\xi_2)^\alpha} \\
|f|_{C_x^\alpha(S_T)} &= \operatorname*{supess}_{t<T} \sup_{x_1,x_2 \in \mathbb{R}^N, x_1 \neq x_2} \frac{|f(x_1,t) - f(x_2,t)|}{\|x_1 - x_2\|^\alpha}. \\
\|f\|_{C_x^\alpha(S_T)} &= |f|_{C_x^\alpha(S_T)} + \|f\|_{L^\infty(S_T)}; \\
\|f\|_{C^\alpha(S_T)} &= |f|_{C^\alpha(S_T)} + \|f\|_{C^0(S_T)}
\end{aligned}
$$

- Our last assumption is:
- (H3) The coefficients $a_{ij} \in C_x^\alpha(\mathbb{R}^{N+1})$ for some $\alpha \in (0,1)$, that is:

$$
|a_{ij}|_{C_x^\alpha(\mathbb{R}^{N+1})} = \operatorname*{supess}_{t\in\mathbb{R}} \sup_{x_1,x_2 \in \mathbb{R}^N, x_1 \neq x_2} \frac{|a_{ij}(x_1,t) - a_{ij}(x_2,t)|}{\|x_1 - x_2\|^\alpha} \leq \Lambda < \infty.
$$

An example of our setting

- For the Kolmogorov type operator

$$\begin{aligned} \mathcal{L}u &= a(x_1, x_2, t)\, u_{x_1 x_1} + x_1 u_{x_2} - u_t \text{ with } (x_1, x_2) \in \mathbb{R}^2 \\ 0 &< \nu \le a(x_1, x_2, t) \le \nu^{-1} \end{aligned}$$

the distance is:

$$\begin{aligned} & d((x_1, x_2, t), (y_1, y_2, s)) \\ = \; & |x_1 - y_1| + |x_2 - y_2 + (t - s)\, y_1|^{1/3} + |t - s|^{1/2}. \end{aligned}$$

An example of our setting

- For the Kolmogorov type operator

$$
\begin{aligned}
\mathcal{L}u &= a(x_1, x_2, t)\, u_{x_1 x_1} + x_1 u_{x_2} - u_t \text{ with } (x_1, x_2) \in \mathbb{R}^2 \\
0 &< \nu \leq a(x_1, x_2, t) \leq \nu^{-1}
\end{aligned}
$$

the distance is:

$$
\begin{aligned}
& d\left((x_1, x_2, t), (y_1, y_2, s)\right) \\
= \;& |x_1 - y_1| + |x_2 - y_2 + (t - s)\, y_1|^{1/3} + |t - s|^{1/2}.
\end{aligned}
$$

- In particular,

$$
d\left((x_1, x_2, t), (y_1, y_2, t)\right) = |x_1 - y_1| + |x_2 - y_2|^{1/3}.
$$

Theorem (Biagi, B.)

Under the above assumptions (H1), (H2), (H3),

1. $\forall T > 0\ \exists c > 0$, *depending on* T, α, B, ν, Λ, *such that*

$$\sum_{i,j=1}^{q} \|\partial^2_{x_i x_j} u\|_{C^\alpha_x(S_T)} + \|Yu\|_{C^\alpha_x(S_T)} + \sum_{i=1}^{q} \|\partial_{x_i} u\|_{C^\alpha(S_T)} + \|u\|_{C^\alpha(S_T)} \leq c \left\{ \|\mathcal{L}u\|_{C^\alpha_x(S_T)} + \|u\|_{C^0(S_T)} \right\}.$$

$\forall u \in \mathcal{S}^\alpha(S_T)$.

Theorem (Biagi, B.)

Under the above assumptions (H1), (H2), (H3),

1. $\forall T > 0\ \exists c > 0$, *depending on* T, α, B, ν, Λ, *such that*

$$\sum_{i,j=1}^{q} \|\partial^2_{x_i x_j} u\|_{C^\alpha_x(S_T)} + \|Yu\|_{C^\alpha_x(S_T)} + \sum_{i=1}^{q} \|\partial_{x_i} u\|_{C^\alpha(S_T)} + \|u\|_{C^\alpha(S_T)} \leq c \left\{ \|\mathcal{L}u\|_{C^\alpha_x(S_T)} + \|u\|_{C^0(S_T)} \right\}.$$

   $\forall u \in \mathcal{S}^\alpha(S_T)$.

2. $\forall T > \tau > -\infty$ *and compact set* $K \subset \mathbb{R}^N$, $\exists c > 0$, *depending on* $K, \tau, T, \alpha, B, \nu, \Lambda$, *such that* $\forall \xi = (x,t)$, $\eta = (y,s) \in K \times [\tau, T]$

$$|\partial^2_{x_i x_j} u(\xi) - \partial^2_{x_i x_j} u(\eta)| \leq c \left\{ \|\mathcal{L}u\|_{C^\alpha_x(S_T)} + \|u\|_{C^0(S_T)} \right\} \left(d(\xi,\eta)^\alpha + |t-s|^{\alpha/q_N} \right).$$

   Here $q_N \geqslant 3$ *is the largest exponent in the dilations* $D(\lambda)$.

- The function space $\mathcal{S}^{\alpha}(S_T)$ of admissible functions u is the set of $u : \overline{S_T} \rightarrow \mathbb{R}$ such that:

- The function space $\mathcal{S}^{\alpha}(S_T)$ of admissible functions u is the set of $u : \overline{S_T} \to \mathbb{R}$ such that:
- $u \in C(\overline{S_T}) \cap L^{\infty}(S_T)$;

- The function space $\mathcal{S}^{\alpha}(S_T)$ of admissible functions u is the set of $u : \overline{S_T} \to \mathbb{R}$ such that:
- $u \in C(\overline{S_T}) \cap L^{\infty}(S_T)$;
- the distributional derivatives $Yu, u_{x_i x_j}$ $(i, j = 1, 2, ..., q)$ belong to $C_x^{\alpha}(S_T)$.

- The function space $\mathcal{S}^{\alpha}(S_T)$ of admissible functions u is the set of $u : \overline{S_T} \to \mathbb{R}$ such that:
- $u \in C(\overline{S_T}) \cap L^{\infty}(S_T)$;
- the distributional derivatives $Yu, u_{x_i x_j}$ $(i, j = 1, 2, ..., q)$ belong to $C_x^{\alpha}(S_T)$.
- We prove that for every $u \in \mathcal{S}^{\alpha}(S_T)$ actually one has:

$$u \text{ and } u_{x_i} \ (i = 1, 2, ..., q) \text{ are } C^{\alpha}(S_T)$$

$$u_{x_i x_j} \text{ are locally Hölder continuous.}$$

- The function space $\mathcal{S}^{\alpha}(S_T)$ of admissible functions u is the set of $u : \overline{S_T} \to \mathbb{R}$ such that:
- $u \in C(\overline{S_T}) \cap L^{\infty}(S_T)$;
- the distributional derivatives $Yu, u_{x_i x_j}$ $(i, j = 1, 2, ..., q)$ belong to $C_x^{\alpha}(S_T)$.
- We prove that for every $u \in \mathcal{S}^{\alpha}(S_T)$ actually one has:

$$u \text{ and } u_{x_i} \ (i = 1, 2, ..., q) \text{ are } C^{\alpha}(S_T)$$

$$u_{x_i x_j} \text{ are locally Hölder continuous.}$$

- Hence, both u_{x_i} and $u_{x_i x_j}$ are continuous in the joint variables;

- The function space $\mathcal{S}^{\alpha}(S_T)$ of admissible functions u is the set of $u : \overline{S_T} \to \mathbb{R}$ such that:
- $u \in C(\overline{S_T}) \cap L^{\infty}(S_T)$;
- the distributional derivatives $Yu, u_{x_i x_j}$ $(i, j = 1, 2, ..., q)$ belong to $C_x^{\alpha}(S_T)$.
- We prove that for every $u \in \mathcal{S}^{\alpha}(S_T)$ actually one has:

$$u \text{ and } u_{x_i} \ (i = 1, 2, ..., q) \text{ are } C^{\alpha}(S_T)$$
$$u_{x_i x_j} \text{ are locally Hölder continuous.}$$

- Hence, both u_{x_i} and $u_{x_i x_j}$ are continuous in the joint variables;
- instead, $Yu \in C_x^{\alpha}(S_T)$ but it may be discontinuous in time;

- The function space $\mathcal{S}^{\alpha}(S_T)$ of admissible functions u is the set of $u : \overline{S_T} \to \mathbb{R}$ such that:
- $u \in C(\overline{S_T}) \cap L^{\infty}(S_T)$;
- the distributional derivatives $Yu, u_{x_i x_j}$ $(i, j = 1, 2, ..., q)$ belong to $C_x^{\alpha}(S_T)$.
- We prove that for every $u \in \mathcal{S}^{\alpha}(S_T)$ actually one has:

$$u \text{ and } u_{x_i} \ (i = 1, 2, ..., q) \text{ are } C^{\alpha}(S_T)$$

$$u_{x_i x_j} \text{ are locally Hölder continuous.}$$

- Hence, both u_{x_i} and $u_{x_i x_j}$ are continuous in the joint variables;
- instead, $Yu \in C_x^{\alpha}(S_T)$ but it may be discontinuous in time;
- For instance:

$$\mathcal{L}u = f(t)$$

with $f \in L^{\infty}(-\infty, T)$ discontinuous, and u only depending on t, bounded and Lipschitz continuous.

- The function space $\mathcal{S}^{\alpha}(S_T)$ of admissible functions u is the set of $u : \overline{S_T} \to \mathbb{R}$ such that:
- $u \in C(\overline{S_T}) \cap L^{\infty}(S_T)$;
- the distributional derivatives Yu, $u_{x_i x_j}$ $(i, j = 1, 2, ..., q)$ belong to $C_x^{\alpha}(S_T)$.
- We prove that for every $u \in \mathcal{S}^{\alpha}(S_T)$ actually one has:

$$u \text{ and } u_{x_i} \ \ (i = 1, 2, ..., q) \text{ are } C^{\alpha}(S_T)$$
$$u_{x_i x_j} \text{ are locally Hölder continuous.}$$

- Hence, both u_{x_i} and $u_{x_i x_j}$ are continuous in the joint variables;
- instead, $Yu \in C_x^{\alpha}(S_T)$ but it may be discontinuous in time;
- For instance:

$$\mathcal{L}u = f(t)$$

  with $f \in L^{\infty}(-\infty, T)$ discontinuous, and u only depending on t, bounded and Lipschitz continuous.
- Point (2) of our theorem is consistent with Knerr's result on parabolic equations on bounded cylinders.

Ashes on my head

- Partial Schauder estimates for degenerate KFP operators have been proved in the recent paper by Chaudru de Raynal, Honoré, Menozzi (Ann. SNS Pisa, 2021)

Ashes on my head

- Partial Schauder estimates for degenerate KFP operators have been proved in the recent paper by Chaudru de Raynal, Honoré, Menozzi (Ann. SNS Pisa, 2021)
- (with different techniques and without getting the Hölder control in time of second order derivatives).

Ashes on my head

- Partial Schauder estimates for degenerate KFP operators have been proved in the recent paper by Chaudru de Raynal, Honoré, Menozzi (Ann. SNS Pisa, 2021)
- (with different techniques and without getting the Hölder control in time of second order derivatives).
- *Point (1) of our theorem is contained in the results of the above paper.*

Ashes on my head

- Partial Schauder estimates for degenerate KFP operators have been proved in the recent paper by Chaudru de Raynal, Honoré, Menozzi (Ann. SNS Pisa, 2021)
- (with different techniques and without getting the Hölder control in time of second order derivatives).
- *Point (1) of our theorem is contained in the results of the above paper.*
- This subject seems to be hot...

Ashes on my head

- Partial Schauder estimates for degenerate KFP operators have been proved in the recent paper by Chaudru de Raynal, Honoré, Menozzi (Ann. SNS Pisa, 2021)
- (with different techniques and without getting the Hölder control in time of second order derivatives).
- *Point (1) of our theorem is contained in the results of the above paper.*
- This subject seems to be hot...
- two recent preprints on arxiv.org dealing with degenerate KFP operators with coefficients Hölder continuous in space and L^∞ in time:

Ashes on my head

- Partial Schauder estimates for degenerate KFP operators have been proved in the recent paper by Chaudru de Raynal, Honoré, Menozzi (Ann. SNS Pisa, 2021)
- (with different techniques and without getting the Hölder control in time of second order derivatives).
- *Point (1) of our theorem is contained in the results of the above paper.*
- This subject seems to be hot...
- two recent preprints on arxiv.org dealing with degenerate KFP operators with coefficients Hölder continuous in space and L^∞ in time:
- Lucertini, Pagliarani, Pascucci: construction of a fundamental solution for these operators, with consequent results about the Cauchy problem;

Ashes on my head

- Partial Schauder estimates for degenerate KFP operators have been proved in the recent paper by Chaudru de Raynal, Honoré, Menozzi (Ann. SNS Pisa, 2021)
- (with different techniques and without getting the Hölder control in time of second order derivatives).
- *Point (1) of our theorem is contained in the results of the above paper.*
- This subject seems to be hot...
- two recent preprints on arxiv.org dealing with degenerate KFP operators with coefficients Hölder continuous in space and L^∞ in time:
- Lucertini, Pagliarani, Pascucci: construction of a fundamental solution for these operators, with consequent results about the Cauchy problem;
- Henderson, Wang: partial Schauder estimates for a special class of KFP operators, with applications to the Landau equation.

Our strategy

- To prove partial Schauder estimates, we follow as close as possible the classical approach to standard Schauder estimates:

Our strategy

- To prove partial Schauder estimates, we follow as close as possible the classical approach to standard Schauder estimates:
- classically, one freezes the coefficients $a_{ij}(x,t)$ at some point $(\overline{x},\overline{t})$ and writes down the representation formula for $u_{x_i x_j}$ in terms of $\mathcal{L}_{(\overline{x},\overline{t})}u$ exploiting the fundamental solution $\Gamma^{(\overline{x},\overline{t})}$ of the frozen operator $\mathcal{L}_{(\overline{x},\overline{t})}$:

$$u_{x_i x_j}(x,t) = -\,\mathrm{PV}\int_{-\infty}^{t}\int_{\mathbb{R}^N}\Gamma_{x_i x_j}^{(\overline{x},\overline{t})}(x,t;y,s)\,\mathcal{L}_{(\overline{x},\overline{t})}u(y,s)\,dyds$$

Our strategy

- To prove partial Schauder estimates, we follow as close as possible the classical approach to standard Schauder estimates:
- classically, one freezes the coefficients $a_{ij}(x,t)$ at some point $(\overline{x},\overline{t})$ and writes down the representation formula for $u_{x_i x_j}$ in terms of $\mathcal{L}_{(\overline{x},\overline{t})}u$ exploiting the fundamental solution $\Gamma^{(\overline{x},\overline{t})}$ of the frozen operator $\mathcal{L}_{(\overline{x},\overline{t})}$:

$$u_{x_i x_j}(x,t) = -\operatorname{PV}\int_{-\infty}^{t}\int_{\mathbb{R}^N}\Gamma_{x_i x_j}^{(\overline{x},\overline{t})}(x,t;y,s)\,\mathcal{L}_{(\overline{x},\overline{t})}u(y,s)\,dyds$$

- then writes

$$\mathcal{L}_{(\overline{x},\overline{t})}u = \mathcal{L}u + \sum\left[a_{ij}(\overline{x},\overline{t}) - a_{ij}(x,t)\right]u_{x_i x_j} \equiv \mathcal{L}u + (\text{remainder})$$

and, if u has small compact support, Hölder continuity of the coefficients makes the remainder small.

Our strategy

- To prove partial Schauder estimates, we follow as close as possible the classical approach to standard Schauder estimates:
- classically, one freezes the coefficients $a_{ij}(x,t)$ at some point $(\overline{x},\overline{t})$ and writes down the representation formula for $u_{x_i x_j}$ in terms of $\mathcal{L}_{(\overline{x},\overline{t})}u$ exploiting the fundamental solution $\Gamma^{(\overline{x},\overline{t})}$ of the frozen operator $\mathcal{L}_{(\overline{x},\overline{t})}$:

$$u_{x_i x_j}(x,t) = -\mathrm{PV}\int_{-\infty}^{t}\int_{\mathbb{R}^N}\Gamma^{(\overline{x},\overline{t})}_{x_i x_j}(x,t;y,s)\,\mathcal{L}_{(\overline{x},\overline{t})}u(y,s)\,dyds$$

- then writes

$$\mathcal{L}_{(\overline{x},\overline{t})}u = \mathcal{L}u + \sum\left[a_{ij}(\overline{x},\overline{t}) - a_{ij}(x,t)\right]u_{x_i x_j} \equiv \mathcal{L}u + (\text{remainder})$$

  and, if u has small compact support, Hölder continuity of the coefficients makes the remainder small.
- In our situation we can exploit only Hölder continuity w.r.t. x, so we can freeze only the x variable.

The operator with coefficients only depending on time

- Hence our model operator is the one with coefficients $a_{ij}(\overline{x}, t)$, i.e. *a KFP operator with (measurable) coefficients only depending on* t.

The operator with coefficients only depending on time

- Hence our model operator is the one with coefficients $a_{ij}(\overline{x}, t)$, i.e. *a KFP operator with (measurable) coefficients only depending on t.*
- The main tool we need is the fundamental solution of a KFP operator of the kind

$$
\begin{aligned}
Lu &= \sum_{i,j=1}^{q} a_{ij}(t)\,\partial^2_{x_i x_j} u + \sum_{k,j=1}^{N} b_{jk} x_k \partial_{x_j} u - \partial_t u, \text{ with } a_{ij} \in L^\infty(\mathbb{R}) \\
& \sum_{i,j=1}^{q} a_{ij}(t)\,\xi_i \xi_j \geq \nu |\xi|^2 \ \ \forall \xi \in \mathbb{R}^q, \text{ a.e. } t \in \mathbb{R}
\end{aligned}
$$

which has been explicitly built and studied in our previous paper [B.-Polidoro, 2020].

Fundamental solution of the operator with coefficients only depending on time

Theorem (B.-Polidoro)

Under assumptions (H1)-(H2), let $C(t,s)$ be the $N \times N$ matrix defined as

$$C(t,s) = \int_s^t E(t-\sigma) \cdot \begin{pmatrix} A_0(\sigma) & 0 \\ 0 & 0 \end{pmatrix} \cdot E(t-\sigma)^T \, d\sigma \quad (\textit{with } t > s)$$

(with $E(\sigma) = \exp(-\sigma B)$). Then $C(t,s)$ is symmetric and positive definite for every $t > s$ and the fundamental solution for L with pole at (y,s) is:

$$\Gamma(x,t;y,s) = \frac{1}{(4\pi)^{N/2}\sqrt{\det C(t,s)}} e^{-\frac{1}{4}\langle C(t,s)^{-1}(x-E(t-s)y),\, x-E(t-s)y\rangle} \cdot \mathbf{1}_{\{t>s\}}$$

(where $\mathbf{1}_A$ denotes the indicator function of a set A).

Fundamental solution of the operator with coefficients only depending on time

- For this Γ, many properties are proved (regularity, limits at infinity, Cauchy problem...).

Fundamental solution of the operator with coefficients only depending on time

- For this Γ, many properties are proved (regularity, limits at infinity, Cauchy problem...).
- In particular, Γ is smooth in x, y (outside the pole) and just locally Lipschitz continuous in t, s.

Fundamental solution of the operator with coefficients only depending on time

- For this Γ, many properties are proved (regularity, limits at infinity, Cauchy problem...).
- In particular, Γ is smooth in x, y (outside the pole) and just locally Lipschitz continuous in t, s.
- The following comparison theorem is also proved in [B.-Polidoro 2020]. Let

$$L_\alpha u = \alpha \sum_{i=1}^{q} \partial^2_{x_i x_i} u + \sum_{k,j=1}^{N} b_{jk} x_k \partial_{x_j} u - \partial_t u,$$

Fundamental solution of the operator with coefficients only depending on time

- For this Γ, many properties are proved (regularity, limits at infinity, Cauchy problem...).
- In particular, Γ is smooth in x, y (outside the pole) and just locally Lipschitz continuous in t, s.
- The following comparison theorem is also proved in [B.-Polidoro 2020]. Let

$$L_\alpha u = \alpha \sum_{i=1}^{q} \partial^2_{x_i x_i} u + \sum_{k,j=1}^{N} b_{jk} x_k \partial_{x_j} u - \partial_t u,$$

- Then the fundamental solution Γ_α of L_α takes the simpler form:

$$\Gamma_\alpha(x, t; 0, 0) = \frac{1}{(4\pi\alpha)^{\frac{N}{2}} t^{\frac{Q}{2}} \sqrt{\det C_0(1)}} e^{-\frac{1}{4\alpha}\left\langle C_0(1)^{-1} D_0\left(\frac{1}{\sqrt{t}}\right)x,\, D_0\left(\frac{1}{\sqrt{t}}\right)x\right\rangle}$$

Fundamental solution of the operator with coefficients only depending on time

- For this Γ, many properties are proved (regularity, limits at infinity, Cauchy problem...).
- In particular, Γ is smooth in x, y (outside the pole) and just locally Lipschitz continuous in t, s.
- The following comparison theorem is also proved in [B.-Polidoro 2020]. Let

$$L_\alpha u = \alpha \sum_{i=1}^{q} \partial^2_{x_i x_i} u + \sum_{k,j=1}^{N} b_{jk} x_k \partial_{x_j} u - \partial_t u,$$

- Then the fundamental solution Γ_α of L_α takes the simpler form:

$$\Gamma_\alpha(x, t; 0, 0) = \frac{1}{(4\pi\alpha)^{\frac{N}{2}} t^{\frac{Q}{2}} \sqrt{\det C_0(1)}} e^{-\frac{1}{4\alpha}\left\langle C_0(1)^{-1} D_0\left(\frac{1}{\sqrt{t}}\right)x,\, D_0\left(\frac{1}{\sqrt{t}}\right)x\right\rangle}$$

- and the following comparison holds:

$$\nu^N \Gamma_\nu(x, t; y, s) \le \Gamma(x, t; y, s) \le \frac{1}{\nu^N} \Gamma_{\nu^{-1}}(x, t; y, s).$$

The steps of the proof of partial Schauder estimates

- The proof of Schauder estimates in [Biagi-B.] is divided into two parts:

The steps of the proof of partial Schauder estimates

- The proof of Schauder estimates in [Biagi-B.] is divided into two parts:
- **Part 1:** we study the model operator with measurable coefficients $a_{ij}(t)$;

The steps of the proof of partial Schauder estimates

- The proof of Schauder estimates in [Biagi-B.] is divided into two parts:
- **Part 1:** we study the model operator with measurable coefficients $a_{ij}(t)$;
- **Part 2:** we study of the operator with coefficients $a_{ij}(x,t) \in C_x^{\alpha}(S_T)$.

Part 1. Operator with coefficients only depending on time

- First of all, we prove *sharp estimates on the space derivatives of every order of the fundamental solution* Γ of the operator with coefficients $a_{ij}(t)$:

Part 1. Operator with coefficients only depending on time

- First of all, we prove *sharp estimates on the space derivatives of every order of the fundamental solution* Γ of the operator with coefficients $a_{ij}(t)$:

Theorem

$\forall$ multiindices α_1, α_2 there exists $c > 0$ such that:

$$\begin{aligned} \left| D_x^{\alpha_1} D_y^{\alpha_2} \Gamma(x, t; y, s) \right| &\leq \frac{c}{(t-s)^{\omega(\alpha_1,\alpha_2)/2}} \Gamma_{c_1 \nu^{-1}}(x, t; y, s) \\ &\leq \frac{c}{d((x,t),(y,s))^{Q+\omega(\alpha_1,\alpha_2)}} \end{aligned}$$

where $\omega(\alpha_1, \alpha_2)$ is the total weight of the multiindex, w.r.t. the homogeneities of the variables.

Part 1. Operator with coefficients only depending on time

- Second, we prove representation formulas for u and $u_{x_i x_j}$:

Part 1. Operator with coefficients only depending on time

- Second, we prove representation formulas for u and $u_{x_i x_j}$:

Theorem

Let $T > \tau > -\infty$. Then $\forall u \in \mathcal{S}^{\alpha}(S_T)$, u vanishing for $t \leq \tau$,

$$u(x,t) = -\int_{\mathbb{R}^N \times (\tau,t)} \Gamma(x,t;y,s) Lu(y,s)\, dy\, ds \quad \forall (x,t) \in S_T$$

Moreover, for $1 \leq i,j \leq q$

$$u_{x_i x_j}(x,t) = \int_{\mathbb{R}^N \times (\tau,t)} \Gamma_{x_i x_j}(x,t;y,s) \left\{ Lu(E(s-t)x,s) - Lu(y,s) \right\} dy\, ds.$$

Part 1. Operator with coefficients only depending on time

- Second, we prove representation formulas for u and $u_{x_i x_j}$:

Theorem

Let $T > \tau > -\infty$. Then $\forall u \in \mathcal{S}^{\alpha}(S_T)$, u vanishing for $t \leq \tau$,

$$u(x,t) = -\int_{\mathbb{R}^N \times (\tau,t)} \Gamma(x,t;y,s) Lu(y,s)\, dy\, ds \quad \forall (x,t) \in S_T$$

Moreover, for $1 \leq i,j \leq q$

$$u_{x_i x_j}(x,t) = \int_{\mathbb{R}^N \times (\tau,t)} \Gamma_{x_i x_j}(x,t;y,s) \left\{ Lu(E(s-t)x,s) - Lu(y,s) \right\} dy\, ds.$$

- The last formula is meaningful in view of the following estimate

$$\int_{\mathbb{R}^N \times (\tau,t)} \left| \Gamma_{x_i x_j}(x,t;y,s) \right| \cdot \left\| E(s-t)x - y \right\|^{\alpha} dy\, ds \leq c(t-\tau)^{\alpha/2}$$

$\forall t > \tau.$

Part 1. Operator with coefficients only depending on time

- Third, we study the singular integral operator with kernel $\Gamma_{x_i x_j}$ over the space $C_x^\alpha(S_T)$:

Part 1. Operator with coefficients only depending on time

- Third, we study the singular integral operator with kernel $\Gamma_{x_i x_j}$ over the space $C_x^\alpha(S_T)$:

Theorem (Hölder continuity of singular integrals)

For $T > \tau > -\infty$, let

$$C_x^\alpha(\tau; T) := \{f \in C_x^\alpha(S_T) : f(x,t) = 0 \text{ for } t \le \tau\}.$$

Then, the linear operator T_{ij} on $C_x^\alpha(\tau; T)$ defined by

$$T_{ij} f(x,t) := \int_{\mathbb{R}^N \times (\tau,t)} \Gamma_{x_i x_j}(x,t;y,s) \{f(E(s-t)x, s) - f(y,s)\}\, dy\, ds$$

satisfies

$$\| T_{ij} f \|_{C_x^\alpha(S_T)} \le c\, |f|_{C_x^\alpha(S_T)} \quad \forall f \in C_x^\alpha(\tau; T)$$

with constant c depending on $(T - \tau)$ and α.

Part 1. Operator with coefficients only depending on time

- The proof of the previous singular integral estimate relies on further properties of the kernel $\Gamma_{x_i x_j}$ which are previously established:

Part 1. Operator with coefficients only depending on time

- The proof of the previous singular integral estimate relies on further properties of the kernel $\Gamma_{x_i x_j}$ which are previously established:

Theorem (Mean value inequality for singular kernels)

There exist $c > 0$, $c_1 > 1$ *such that*

$$\left|\Gamma_{x_i x_j}(\xi_1, \eta) - \Gamma_{x_i x_j}(\xi_2, \eta)\right| \leq c \frac{d(\xi_1, \xi_2)}{d(\xi_1, \eta)^{Q+3}}$$

$\forall \xi_1 = (x_1, t_1), \xi_2 = (x_2, t_2) \in \mathbb{R}^{N+1}$ *such that*

$$d(\xi_1, \eta) \geq c_1 d(\xi_1, \xi_2) > 0.$$

Part 1. Operator with coefficients only depending on time

- The proof of the previous singular integral estimate relies on further properties of the kernel $\Gamma_{x_i x_j}$ which are previously established:

Theorem (Mean value inequality for singular kernels)

There exist $c > 0$, $c_1 > 1$ *such that*

$$\left|\Gamma_{x_i x_j}(\xi_1, \eta) - \Gamma_{x_i x_j}(\xi_2, \eta)\right| \leq c \frac{d(\xi_1, \xi_2)}{d(\xi_1, \eta)^{Q+3}}$$

$\forall \xi_1 = (x_1, t_1), \xi_2 = (x_2, t_2) \in \mathbb{R}^{N+1}$ *such that*

$$d(\xi_1, \eta) \geq c_1 d(\xi_1, \xi_2) > 0.$$

- The above result depends on the sharp estimates on the derivatives of Γ previously established.

Part 1. Operator with coefficients only depending on time

- The second result on the kernel $\Gamma_{x_i x_j}$ which allows to prove the singular integral estimate is the following:

Part 1. Operator with coefficients only depending on time

- The second result on the kernel $\Gamma_{x_i x_j}$ which allows to prove the singular integral estimate is the following:

Theorem (Cancellation property)

$\exists c > 0$ such that, for $1 \leq i, j \leq q$, one has:

$$\int_{\tau}^{t} \left| \int_{\{y \in \mathbb{R}^N : d((x,t),(y,s)) \geq r\}} \Gamma_{x_i x_j}(x, t; y, s)\, dy \right| ds \leq c,$$

$\forall x \in \mathbb{R}^N$, $\tau < t$ and $r > 0$.

Part 1. Operator with coefficients only depending on time

- Now, representation formulas on $u_{x_i x_j}$ plus continuity of the singular integral on $C_x^{\alpha}(S_T)$ gives the first step of Schauder estimates:

Part 1. Operator with coefficients only depending on time

- Now, representation formulas on $u_{x_i x_j}$ plus continuity of the singular integral on $C_x^\alpha(S_T)$ gives the first step of Schauder estimates:

Theorem

Let $T > \tau > -\infty$ and $\alpha \in (0,1)$. Then, $\exists c > 0$, only depending on $(T-\tau), \alpha, \nu, B$, such that

$$\sum_{i,j=1}^{q} \left\| u_{x_i x_j} \right\|_{C_x^\alpha(S_T)} \leq c \left| Lu \right|_{C_x^\alpha(S_T)}$$

$$\left\| Yu \right\|_{C_x^\alpha(S_T)} \leq c \left\| Lu \right\|_{C_x^\alpha(S_T)},$$

$\forall u \in \mathcal{S}^\alpha(S_T)$ vanishing for $t \leq \tau$.

Part 1. Operator with coefficients only depending on time

- Now, representation formulas on $u_{x_i x_j}$ plus continuity of the singular integral on $C_x^\alpha(S_T)$ gives the first step of Schauder estimates:

Theorem

Let $T > \tau > -\infty$ and $\alpha \in (0,1)$. Then, $\exists c > 0$, only depending on $(T-\tau), \alpha, \nu, B$, such that

$$\sum_{i,j=1}^{q} \|u_{x_i x_j}\|_{C_x^\alpha(S_T)} \leq c\, |Lu|_{C_x^\alpha(S_T)}$$

$$\|Yu\|_{C_x^\alpha(S_T)} \leq c\, \|Lu\|_{C_x^\alpha(S_T)},$$

$\forall u \in \mathcal{S}^\alpha(S_T)$ vanishing for $t \leq \tau$.

- Recall that, so far, L is the operator with coefficients $a_{ij}(t)$.

Part 1. Operator with coefficients only depending on time

- Our second result for operators with measurable coefficients $a_{ij}(t)$ is the following:

Part 1. Operator with coefficients only depending on time

- Our second result for operators with measurable coefficients $a_{ij}(t)$ is the following:

Theorem

Let $T > \tau > -\infty, \alpha \in (0,1)$ *and let* $K \subseteq \mathbb{R}^N$ *be a compact set.* $\exists c = c(K, \tau, T) > 0$ *such that,* $\forall u \in \mathcal{S}^{\alpha}(S_T)$ *vanishing for* $t \leq \tau$,

$$\begin{aligned} & \left| u_{x_i x_j}(x_1, t_1) - u_{x_i x_j}(x_2, t_2) \right| \\ \leq \; & c \, |Lu|_{C_x^{\alpha}(S_T)} \left\{ d((x_1, t_1), (x_2, t_2))^{\alpha} + |t_1 - t_2|^{\alpha / q_N} \right\} \end{aligned}$$

for $1 \leq i, j \leq q, \forall (x_1, t_1), (x_2, t_2) \in K \times [\tau, T]$.

Part 1. Operator with coefficients only depending on time

- Our second result for operators with measurable coefficients $a_{ij}(t)$ is the following:

Theorem

Let $T > \tau > -\infty, \alpha \in (0,1)$ and let $K \subseteq \mathbb{R}^N$ be a compact set. $\exists c = c(K, \tau, T) > 0$ such that, $\forall u \in \mathcal{S}^\alpha(S_T)$ vanishing for $t \le \tau$,

$$\begin{aligned} & \left| u_{x_i x_j}(x_1, t_1) - u_{x_i x_j}(x_2, t_2) \right| \\ \le \; & c \left| Lu \right|_{C_x^\alpha(S_T)} \left\{ d((x_1, t_1), (x_2, t_2))^\alpha + |t_1 - t_2|^{\alpha/q_N} \right\} \end{aligned}$$

for $1 \le i, j \le q, \forall (x_1, t_1), (x_2, t_2) \in K \times [\tau, T]$.

- Recall that $q_N \ge 3$ is the largest exponent in the dilations $D_0(\lambda)$.

Part 1. Operator with coefficients only depending on time

- To prove the above theorem, we exploit and revise the proof of the singular integral continuity on the space $C_x^{\alpha}(S_T)$.

Part 1. Operator with coefficients only depending on time

- To prove the above theorem, we exploit and revise the proof of the singular integral continuity on the space $C_x^\alpha(S_T)$.
- When we bound the increment of $u_{x_i x_j}$ corresponding to different times t_1, t_2, recalling the representation formula

$$u_{x_i x_j}(x,t) = \int_\tau^t \int_{\mathbb{R}^N} \Gamma_{x_i x_j}(x,t;y,s) \{Lu(E(s-t)x,s) - Lu(y,s)\}\, dyds$$

we are lead to bound the quantities $\|x - E(t)x\|, \|(E(t) - E(s))x\|$.

Part 1. Operator with coefficients only depending on time

- To prove the above theorem, we exploit and revise the proof of the singular integral continuity on the space $C_x^{\alpha}(S_T)$.
- When we bound the increment of $u_{x_i x_j}$ corresponding to different times t_1, t_2, recalling the representation formula

$$u_{x_i x_j}(x,t) = \int_{\tau}^{t} \int_{\mathbb{R}^N} \Gamma_{x_i x_j}(x,t;y,s) \left\{ Lu(E(s-t)x,s) - Lu(y,s) \right\} dyds$$

we are lead to bound the quantities $\|x - E(t)x\|, \|(E(t) - E(s))x\|$.
- Exploiting the properties of the exponential matrix $E(t)$, the best we can prove is:

Part 1. Operator with coefficients only depending on time

- To prove the above theorem, we exploit and revise the proof of the singular integral continuity on the space $C_x^{\alpha}(S_T)$.
- When we bound the increment of $u_{x_i x_j}$ corresponding to different times t_1, t_2, recalling the representation formula

$$u_{x_i x_j}(x,t) = \int_{\tau}^{t} \int_{\mathbb{R}^N} \Gamma_{x_i x_j}(x,t;y,s) \left\{ Lu(E(s-t)x,s) - Lu(y,s) \right\} dyds$$

  we are lead to bound the quantities $\|x - E(t)x\|, \|(E(t) - E(s))x\|$.
- Exploiting the properties of the exponential matrix $E(t)$, the best we can prove is:

Lemma

Let $K \subseteq \mathbb{R}^N$ be a fixed compact set, and let $T > \tau > -\infty$. There exists a constant $c = c(K, \tau, T) > 0$ such that

$$\begin{aligned} \|x - E(t)x\| &\leq c\,|t-s|^{1/q_N} \quad \forall x \in K \text{ and } t \in [\tau, T] \\ \|(E(t) - E(s))x\| &\leq c\,|t-s|^{1/q_N} \quad \forall x \in K \text{ and } t, s \in [\tau, T]. \end{aligned}$$

Part 2. Operator with coefficients depending on x and t

- We now come to the operator $\mathcal{L}$ with coefficients $a_{ij}(x, t)$ belonging to $C_x^{\alpha}\left(\mathbb{R}^{N+1}\right)$.

Part 2. Operator with coefficients depending on x and t

- We now come to the operator $\mathcal{L}$ with coefficients $a_{ij}(x, t)$ belonging to $C_x^\alpha(\mathbb{R}^{N+1})$.
- The procedure is now fairly classical (where the classical operator with constant coefficients is replaced by the operator with measurable coefficients $a_{ij}(t)$, previously studied).

Part 2. Operator with coefficients depending on x and t

- We now come to the operator $\mathcal{L}$ with coefficients $a_{ij}(x,t)$ belonging to $C_x^{\alpha}\left(\mathbb{R}^{N+1}\right)$.
- The procedure is now fairly classical (where the classical operator with constant coefficients is replaced by the operator with measurable coefficients $a_{ij}(t)$, previously studied).
- Starting with the estimates on $\left\|u_{x_i x_j}\right\|_{C_x^{\alpha}(S_T)}$ proved for the operator L with coefficients $a_{ij}(t)$, we establish, by a perturbative argument (classical "Korn's trick"):

Part 2. Operator with coefficients depending on x and t

- We now come to the operator $\mathcal{L}$ with coefficients $a_{ij}(x,t)$ belonging to $C_x^\alpha\left(\mathbb{R}^{N+1}\right)$.
- The procedure is now fairly classical (where the classical operator with constant coefficients is replaced by the operator with measurable coefficients $a_{ij}(t)$, previously studied).
- Starting with the estimates on $\|u_{x_ix_j}\|_{C_x^\alpha(S_T)}$ proved for the operator L with coefficients $a_{ij}(t)$, we establish, by a perturbative argument (classical "Korn's trick"):

Theorem (Schauder estimates for functions with small compact support)

$\exists c, r_0 > 0$ such that $\forall \overline{\xi} \in S_T$, $r \le r_0$ and $u \in \mathcal{S}^\alpha(S_T)$ with $\operatorname{supp}(u) \subseteq B_r(\overline{\xi}) \cap \overline{S_T}$, one has

$$\sum_{i,j=1}^{q} \|u_{x_ix_j}\|_{C_x^\alpha(B_r(\overline{\xi})\cap S_T)} \le c\,|\mathcal{L}u|_{C_x^\alpha(B_r(\overline{\xi})\cap S_T)}.$$

Part 2. Operator with coefficients depending on x and t

- Interpolation inequalities are a typical tool to deduce global estimates starting with local estimates for compactly supported functions. We will need the following:

Part 2. Operator with coefficients depending on x and t

- Interpolation inequalities are a typical tool to deduce global estimates starting with local estimates for compactly supported functions. We will need the following:

Theorem (Interpolation inequality)

$\forall r > 0$ $\exists c > 0$ and $\gamma > 1$ such that $\forall \varepsilon \in (0,1)$, $\overline{\xi} \in S_T$ and $u \in \mathcal{S}^0(S_T)$,

$$\sum_{h=1}^{q} \|u_{x_h}\|_{C^\alpha(B_r^T(\overline{\xi}))} + \|u\|_{C^\alpha(B_r^T(\overline{\xi}))}$$

$$\leq \varepsilon \left\{ \sum_{h,k=1}^{q} \|u_{x_k x_h}\|_{C^0(B_{4r}^T(\overline{\xi}))} + \|Yu\|_{C^0(B_{4r}^T(\overline{\xi}))} \right\} + \frac{c}{\varepsilon^\gamma} \|u\|_{C^0(B_{4r}^T(\overline{\xi}))}.$$

where $B_r^T(\overline{\xi}) = B_r(\overline{\xi}) \cap S^T$.

Part 2. Operator with coefficients depending on x and t

- Combining the local Schauder estimates for functions with small bounded support, with a covering of S_T, suitable cutoff functions and the interpolation inequality we get:

Part 2. Operator with coefficients depending on x and t

- Combining the local Schauder estimates for functions with small bounded support, with a covering of S_T, suitable cutoff functions and the interpolation inequality we get:

Theorem (Global Schauder estimates in space)

$\exists c > 0$, *depending on* T, α, *the matrix* B *and the numbers* ν *and* Λ, *respectively, such that*

$$\sum_{h,k=1}^{q} \|u_{x_h x_k}\|_{C_x^\alpha(S_T)} + \|Yu\|_{C_x^\alpha(S_T)} + \sum_{k=1}^{q} \|u_{x_k}\|_{C^\alpha(S_T)} + \|u\|_{C^\alpha(S_T)} \leq c \left\{ \|\mathcal{L}u\|_{C_x^\alpha(S_T)} + \|u\|_{C^0(S_T)} \right\}$$

for every $u \in \mathcal{S}^\alpha(S_T)$.

Part 2. Operator with coefficients depending on x and t

- To prove local Hölder estimates in (x, t) on $u_{x_i x_j}$ we first apply to the frozen operator $\mathcal{L}_{\overline{x}}$ the result proved for operators with coefficients $a_{ij}(t)$:

$$\begin{aligned} &\left|u_{x_i x_j}(x_1, t_1) - u_{x_i x_j}(x_2, t_2)\right| \\ &\qquad \leq c \left|\mathcal{L}_{\overline{x}} u\right|_{C_x^\alpha\left(B_r^T\left(\overline{\xi}_i\right)\right)} \left\{ d\left((x_1, t_1), (x_2, t_2)\right)^\alpha + \left|t_1 - t_2\right|^{\alpha/q_N} \right\}. \end{aligned}$$

Part 2. Operator with coefficients depending on x and t

- To prove local Hölder estimates in (x, t) on $u_{x_i x_j}$ we first apply to the frozen operator $\mathcal{L}_{\overline{x}}$ the result proved for operators with coefficients $a_{ij}(t)$:

$$\begin{aligned} &\left| u_{x_i x_j}(x_1, t_1) - u_{x_i x_j}(x_2, t_2) \right| \\ &\qquad \leq c \left| \mathcal{L}_{\overline{x}} u \right|_{C_x^\alpha \left(B_r^T(\overline{\xi}_i) \right)} \left\{ d\left((x_1, t_1), (x_2, t_2) \right)^\alpha + \left| t_1 - t_2 \right|^{\alpha / q_N} \right\}. \end{aligned}$$

- Next, we can write:

$$\begin{aligned} \left| \mathcal{L}_{\overline{x}} u \right|_{C_x^\alpha \left(B_r^T(\overline{\xi}_i) \right)} \leq& \left| \mathcal{L} u \right|_{C_x^\alpha \left(B_r^T(\overline{\xi}_i) \right)} \\ &+ \sum_{i,j=1}^{q} \left| \left[a_{ij}(\overline{x}, t) - a_{ij}(\cdot, t) \right] u_{x_i x_j} \right|_{C_x^\alpha \left(B_r^T(\overline{\xi}_i) \right)}. \end{aligned}$$

Part 2. Operator with coefficients depending on x and t

- To prove local Hölder estimates in (x, t) on $u_{x_i x_j}$ we first apply to the frozen operator $\mathcal{L}_{\overline{x}}$ the result proved for operators with coefficients $a_{ij}(t)$:

$$\begin{aligned}&\left|u_{x_i x_j}(x_1, t_1) - u_{x_i x_j}(x_2, t_2)\right| \\ &\qquad \leq c \left|\mathcal{L}_{\overline{x}} u\right|_{C_x^\alpha\left(B_r^T\left(\overline{\xi}_i\right)\right)} \left\{d\left((x_1, t_1), (x_2, t_2)\right)^\alpha + |t_1 - t_2|^{\alpha/q_N}\right\}.\end{aligned}$$

- Next, we can write:

$$\begin{aligned}\left|\mathcal{L}_{\overline{x}} u\right|_{C_x^\alpha\left(B_r^T\left(\overline{\xi}_i\right)\right)} &\leq \left|\mathcal{L} u\right|_{C_x^\alpha\left(B_r^T\left(\overline{\xi}_i\right)\right)} \\ &\quad + \sum_{i,j=1}^{q} \left|\left[a_{ij}(\overline{x}, t) - a_{ij}(\cdot, t)\right] u_{x_i x_j}\right|_{C_x^\alpha\left(B_r^T\left(\overline{\xi}_i\right)\right)}.\end{aligned}$$

- and, if u has small compact support, exploiting the estimate proved for $\left|u_{x_i x_j}\right|_{C_x^\alpha\left(B_r^T\left(\overline{\xi}_i\right)\right)}$ we get the Hölder bound in terms of $\left|\mathcal{L} u\right|_{C_x^\alpha\left(B_r^T\left(\overline{\xi}_i\right)\right)}$.

Part 2. Operator with coefficients depending on x and t

- Using now a covering of $K \times [\tau, T]$ with small balls and exploiting again the global Schauder estimates in space we finally get:

Part 2. Operator with coefficients depending on x and t

- Using now a covering of $K \times [\tau, T]$ with small balls and exploiting again the global Schauder estimates in space we finally get:

Theorem (Hölder continuity of $u_{x_i x_j}$ in time)

$\forall T > \tau > -\infty$, $\forall$ compact set $K \subset \mathbb{R}^N$, $\exists c > 0$ such that, $\forall u \in \mathcal{S}^\alpha (S_T)$ we have:

$$\begin{aligned} & \left| u_{x_i x_j}(x_1, t_1) - u_{x_i x_j}(x_2, t_2) \right| \\ \leq \quad & c \left\{ \|\mathcal{L}u\|_{C_x^\alpha(S_T)} + \|u\|_{C^0(S_T)} \right\} \cdot \left\{ d((x_1, t_1), (x_2, t_2))^\alpha + |t_1 - t_2|^{\alpha/q_N} \right\} \end{aligned}$$

$\forall (x_1, t_1), (x_2, t_2) \in K \times [\tau, T]$.
In particular, even the second derivatives $u_{x_i x_j}$ are jointly continuous in S_T.

Thank you!