# Partial Regularity in Time for the Landau Equation (with Coulomb Interaction)

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**Landau equation** with unknown $f \equiv f(t,v) \geq 0$:

$$\partial_t f(t,v) = \mathrm{div}_v \int_{\mathbf{R}^3} a(v-w)(\nabla_v - \nabla_w)(f(t,v)f(t,w))dw\,, \quad v \in \mathbf{R}^3$$

with the notation:

$$a(z) := \tfrac{1}{8\pi}\nabla^2|z| = \tfrac{1}{8\pi|z|}\Pi(z)\,, \quad \Pi(z) := I - \left(\tfrac{z}{|z|}\right)^{\otimes 2}$$

**Nonconservative form**

$$\partial_t f(t,v) = (a_{ij} \star_v f(t,v))\,\partial_{v_i}\partial_{v_j} f(t,v) + f(t,v)^2$$

**Open question** global existence of classical solutions or finite-time blow-up for the Cauchy problem with $f\big|_{t=0} = f_{in}$?

**Semilinear heat equation** Finite time blow-up for $u \geq 0$ soln of

$$\partial_t u = \Delta_x u + \alpha u^2$$

**Kaplan's method** Ricatti inequality $\dot{L}(t) \geq -\lambda_0 L(t) + \alpha L^2(t)$ for

$$L(t) := \frac{\int_B u(t,x)\phi(x)dx}{\int_B \phi(x)dx} \quad \text{with} \begin{cases} -\Delta\phi = \lambda_0\phi\,, & \phi > 0 \text{ on } B \\ \phi\big|_{\partial B} = 0 \end{cases}$$

**"Isotropic Landau"** global existence of radially symmetric nonincreasing soln [Gressman-Krieger-Strain 2012, Gualdani-Guillen 2016]

$$\partial_t u = ((-\Delta)^{-1}u)\Delta u + \alpha u^2$$

**Conditional regularity** $L^\infty_t L^p_k$ solns with $p > \frac{3}{2}$ and $k > 5$ are $L^\infty_{t,v}$ ([Silvestre 2017], radial solutions [Gualdani-Guillen 2016])

# (Formal) Conservation Laws+H Theorem

(1) **Conservation of mass+momentum+energy**

$$\frac{d}{dt}\int_{\mathbf{R}^3} f(t,v)\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix} dv = 0$$

(2) **H Theorem** Assuming that $f(t,v) > 0$ a.e., one has

$$\frac{d}{dt}\int_{\mathbf{R}^3} f(t,v)\ln f(t,v)dv$$

$$= -\int_{\mathbf{R}^6} \underbrace{\frac{f(t,v)f(t,w)}{16\pi|v-w|}\left|\Pi(v-w)\left(\frac{\nabla_v f(t,v)}{f(t,v)} - \frac{\nabla_w f(t,w)}{f(t,w)}\right)\right|^2}_{=\frac{1}{4\pi}\left|\Pi(v-w)(\nabla_v-\nabla_w)\sqrt{f(t,v)f(t,w)/|v-w|}\right|^2} dvdw$$

# Villani's H-Solutions [ARMA1998]

**Notation** $\|g\|^p_{L^p_k} := \int (1+|v|^2)^{k/2}|g(v)|^p dv$ with $p \geq 1$ and $k \in \mathbf{R}$

**H-solution** $0 \leq f \in C([0,T);\mathcal{D}'(\mathbf{R}^3)) \cap L^1((0,T);L^1_{-1}(\mathbf{R}^3))$ s.t.

$$\begin{cases} \displaystyle\int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f(t,v)dv = \int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f_{in}(v)dv & \\ & \text{a.e. in } t \geq 0 \\ \displaystyle\int_{\mathbf{R}^3} f(t,v)\ln f(t,v)dv \leq \int_{\mathbf{R}^3} f_{in}(v)\ln f_{in}(v)dv & \end{cases}$$

and, for all $\phi \in C^1_c([0,T)\times\mathbf{R}^3)$, with $\Phi(t,v) := \nabla_v \phi(t,v)$, one has

$$\int_{\mathbf{R}^3} f_{in}(v)\phi(0,v)dv + \int_0^T \int_{\mathbf{R}^3} f(t,v)\partial_t\phi(t,v)dv$$

$$= \int_0^T \int_{\mathbf{R}^6} (\Phi(t,v)-\Phi(t,w))\cdot\Pi(v-w)\,(F(\nabla_v - \nabla_w)F)\,(t,v,w)dvdw$$

with $F(t,v,w) := \sqrt{f(t,v)f(t,w)/8\pi|v-w|}$

**THM**

For each $0 \le f \in L^1_2(\mathbf{R}^3)$ s.t. $f \ln f \in L^1(\mathbf{R}^3)$, one has

$$\int_{\mathbf{R}^3} \frac{|\nabla\sqrt{f(v)}|^2 dv}{(1+|v|^2)^{3/2}} \le C_D + C_D \int_{\mathbf{R}^6} \frac{|\Pi(v-w)(\nabla_v - \nabla_w)\sqrt{f(v)f(w)}|^2}{|v-w|} dvdw$$

with

$$C_D \equiv C_D\left[\int_{\mathbf{R}^3} (1, v, |v|^2, |\ln f(v)|) f(v) dv\right] > 0$$

**COROLLARY**

Let $0 \le f_{in} \in L^1_k(\mathbf{R}^3)$ with $k > 2$ s.t. $f_{in}|\ln f_{in}| \in L^1(\mathbf{R}^3)$; then

$$f \text{ H-solution s.t. } f\Big|_{t=0} = f_{in} \implies f \in L^\infty(0, T; L^1_k(\mathbf{R}^3))$$

•The Desvillettes theorem puts the Landau equation in the same class as $3d$ Navier-Stokes in terms of Lebesgue exponents — except for the $(1+|v|)^{-3}$ weight

| | | |
|---|---|---|
| Navier-Stokes | $u \in L^\infty_t L^2_x\,,$ | $\nabla_x u \in L^2_t L^2_x$ |
| Landau | $\sqrt{f} \in L^\infty_t L^2_v\,,$ | $\nabla_v \sqrt{f} \cap L^2_t L^2_{-3}$ |

•For $3d$ Navier-Stokes, Leray (1934) proved that the set of singular times of his "turbulent solutions" is of $\mathcal{H}^{1/2}$-measure 0.

**PBM** What is the maximum Hausdorff dimension of the set of singular times for a "weak" solution of the Landau equation?

# Hausdorff Dimension

Let $(X, d)$ be a metric space; for all $E \subset X$, set $\Omega_r := \frac{\pi^{r/2}}{\Gamma(1+r/2)}$ and

$$\mathcal{H}^r(E) := \Omega_r \sup_{\delta>0} \inf \left\{ \sum_j \operatorname{diam}(E_j)^r \,:\, \operatorname{diam}(E_j) < \delta \,,\; E \subset \bigcup_j E_j \right\}$$

**THM** For $r \geq 0$, the set function $\mathcal{H}^r$ is a Borel measure on $X$. Besides

$$\mathcal{H}^r(E) > 0 \implies \mathcal{H}^s(E) = +\infty \qquad \text{whenever } r > s \geq 0$$

**DEF** For each $E \subset X$, one defines

$$\mathcal{H} - \dim(E) = \inf\{r \geq 0 \text{ s.t. } \mathcal{H}^r(E) = 0\}$$

**Example** $\mathcal{H} - \dim(\text{middle third Cantor set}) = \frac{\ln 2}{\ln 3}$

**DEF** For each $g \equiv g(v) \geq 0$ measurable, the truncated entropy is

$$H_+(g|\kappa) := \int_{\mathbf{R}^3} \kappa h_+\left(\frac{g(v)}{\kappa}\right) dv\,, \qquad h_+(z) := z(\ln z)_+ - (z-1)_+$$

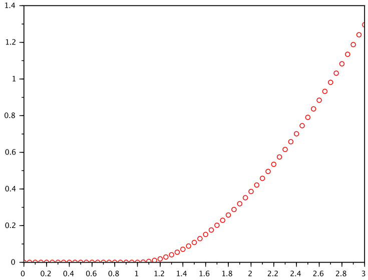


Figure: Graph of the function $h_+$

# (Formal) Truncated H Theorem

One has

$$\frac{d}{dt}H_+(f(t,\cdot)|\kappa)$$

$$+\underbrace{\int \frac{f(t,v)f(t,w)}{16\pi|v-w|}\left|\Pi(v-w)\left(\frac{\mathbf{1}_{f(t,v)>\kappa}\nabla_v f(t,v)}{f(t,v)}-\frac{\mathbf{1}_{f(t,w)>\kappa}\nabla_w f(t,w)}{f(t,w)}\right)\right|^2 dvdw}_{D_1}$$

$$=-\int f(t,v)f(t,w)a(v-w):\nabla_v(\ln \tfrac{f(t,v)}{\kappa})_+\otimes\nabla_w(\ln \tfrac{f(t,w)}{\kappa})_- dvdw$$

$$=-\int a(v-w):\nabla_v f(t,v)\mathbf{1}_{f(t,v)\geq\kappa}\otimes\nabla_w f(t,w)\mathbf{1}_{f(t,w)<\kappa}dvdw$$

$$=\int \underbrace{-\operatorname{div}_v(\operatorname{div}_w a(v-w))}_{\geq 0\text{ ( in fact }=\delta(v-w)\text{ )}}(f(t,v)-\kappa)_+(\kappa-(f(t,w)-\kappa)_-)dvdw$$

$$\leq \underbrace{\kappa\int(f(t,v)-\kappa)_+ dv}_{\textbf{depleted NL}}$$

**DEF** Let $\mathcal{N} \subset [0,\infty)$ be Lebesgue-negligible, let $q \geq 1$ and $C_E > 0$. A $(\mathcal{N}, q, C_E)$-suitable solution on $[0,T) \times \mathbf{R}^3$ is an H-solution s.t., for all $t_1 < t_2 \in [0,T) \setminus \mathcal{N}$ and $\kappa \geq 1$,

$$H_+(f(t_2,\cdot)|\kappa) + C_E \int_{t_1}^{t_2} \left\| \mathbf{1}_{f(t,v)>\kappa} \nabla_v f(t,v)^{1/q} \right\|^2_{L^q(\mathbf{R}^3)} dt$$
$$\leq H_+(f(t_1,\cdot)|\kappa) + 2\kappa \int_{t_1}^{t_2} \int_{\mathbf{R}^3} (f(t,v)-\kappa)_+ dvdt$$

**PROP 1** For each $T > 0$ and all $f_{in} \geq 0$ measurable on $\mathbf{R}^3$ s.t.

$$\int_{\mathbf{R}^3} (1 + |v|^k + |\ln f_{in}(v)|) f_{in}(v) dv < \infty \qquad \text{for some } k > 3$$

there exists an $(\mathcal{N}, q, C_E)$-suitable solution $f$ of the Landau equation on $[0,T]$ with initial data $f_{in}$ and

$$C_E \equiv C_E[T,q,f_{in}] > 0\,, \qquad q := \frac{2k}{k+3}$$

**DEF** A regular time of $f$, suitable solution on $I \subset (0, +\infty)$, is a time $\tau \in I$ s.t. $f \in L^\infty((\tau - \epsilon, \tau) \times \mathbf{R}^3)$ for some $\epsilon \in (0, \tau)$.

The set of singular (i.e. nonregular) times of $f$ on $I$ is denoted $\mathbf{S}[f, I]$.

**Main THM** Let $f$ be a suitable solution to the Landau equation on $[0, T) \times \mathbf{R}^3$ for all $T > 0$ , with initial data $f_{in}$ satisfying

$$\int_{\mathbf{R}^3} (1 + |v|^k + |\ln f_{in}(v)|) f_{in}(v) dv < \infty \quad \text{for all } k > 3$$

Then

$$\mathcal{H} - \dim(\mathbf{S}[f, (0, +\infty)]) \leq \tfrac{1}{2}$$

# Proof of PROP 1 (Sketch)

• Replace $a$ with its truncated variant

$$a_n(z) = \frac{1}{8\pi}(\frac{1}{|z|} \wedge n)\Pi(z)\,, \qquad \text{satisfying } \operatorname{div}(\operatorname{div} a_n) \geq 0$$

• Use the Desvillettes theorem to bound

$$\frac{1}{C'_D} \int_{\mathbf{R}^3} \frac{|\nabla_v \sqrt{f(t,v)}|^2}{(1+|v|)^3} \mathbf{1}_{f(t,v)>\kappa} dv \leq D_1 + \int_{\mathbf{R}^3} (f(t,w) - \kappa)_+ dw$$

• Using the Desvillettes corollary with $p' = 2/q$ (recall $q \in (1,2)$)

$$\left\| \mathbf{1}_{f(t,v)>\kappa} \nabla_v f(t,v)^{1/q} \right\|^q_{L^q(\mathbf{R}^3)}$$

$$\leq (\tfrac{2}{q})^q \|f(t,\cdot)\|_{L^p_{3p/2p'}(\mathbf{R}^3)} \left( \int_{\mathbf{R}^3} \frac{|\nabla_v \sqrt{f(t,v)}|^2 \mathbf{1}_{f(t,v)\geq\kappa}}{(1+|v|^2)^{3/2}} dv \right)^{1/p'}$$

# The 1st De Giorgi Type Lemma

**PROP 2** Let $f$ be a $(\mathcal{N}, q, C_E)$–suitable solution to the Landau equation for $t \in [0,1]$ with $C_E > 0$ and $q \in (\frac{6}{5}, 2)$

Then there exists $\eta_0 \equiv \eta_0[q, C_E] > 0$ s.t.

$$\int_{1/8}^{1} H_+(f(t,\cdot)|\tfrac{1}{2})dt < \eta_0 \implies f(t,v) \leq 2 \quad \text{a.e. on } [\tfrac{1}{2}, 1] \times \mathbf{R}^3$$

# Proof of Prop 2

Set

$$\begin{cases} t^k := \frac{1}{2} - \frac{1}{4} \cdot 2^{-k}\,, & \kappa_k := (1 + (2^{1/q} - 1)(1 - 2^{-k}))^q \\ f_k^+(t,v) := \mu((f(t,v)^{1/q} - \kappa_k^{1/q})_+) & \text{with } \mu(r) := \min(r, r^2) \end{cases}$$

and observe that

$$c_h \mu(r) \le h_+(r) \le C_\theta (r-1)_+^\theta\,, \qquad \theta > 0$$

Consider the quantity

$$A_k := \operatorname*{ess\,sup}_{t^k \le t \le 1} \frac{c_h}{2} \int_{\mathbf{R}^3} f_k^+(t,v)^q dv + \frac{C_E}{4} \int_{t^k}^1 \left( \int_{\mathbf{R}^3} |\nabla_v f_k^+(t,v)|^q dv \right)^{\frac{2}{q}} dt$$

# De Giorgi Nonlinearization [Mem. Accad. Sci. Torino 1957]

•Observe first that

$$f_{k+1}^{+} > 0 \implies f_k^{+} > \mu((2^{1/q} - 1) \cdot 2^{-k-1})$$

$$\text{so that} \quad A_{k+1} \leq C_{q,\theta} 4^{(k+3)q(1+\theta)} \int_{t^k}^{1} \int_{\mathbf{R}^3} f_k^{+}(\tau, v)^{q(1+\theta)} dv d\tau$$

•Using the Hölder inequality + Sobolev embedding with $\iota = \frac{2}{3}$

$$A_{k+1} \leq M \Lambda^k A_k^{\beta}, \quad \beta := \frac{8}{3} - \frac{2}{q} > 1 \text{ and } \Lambda := 2 \cdot 4^{\frac{5q}{3}}$$

with $M \equiv M[q, C_E] > 0$, so that

$$A_0 < M^{-\frac{1}{\beta-1}} \Lambda^{-\frac{1}{(\beta-1)^2}} \implies A_k \to 0 \text{ as } k \to +\infty$$

•Control $A_0$ by truncated entropy + conclude by Fatou's lemma

**PROP 3** Let $f$ be a $(\mathcal{N}, q, C_E)$-suitable solution to the Landau equation on $[0,1]$ with $q \in (\frac{4}{3}, 2)$. There exists $\eta_1 \equiv \eta_1[q, C_E] > 0$ and $\delta_1 \in (0,1)$ such that

$$\overline{\lim_{\epsilon \to 0^+}} \epsilon^{\gamma-3} \int_{1-\epsilon^\gamma}^{1} \left\| \mathbf{1}_{f(T,V)>\epsilon^{-\gamma}} \nabla_V f(T,V)^{\frac{1}{q}} \right\|_{L^q(\mathbf{R}^3)}^2 dT < \eta_1$$
$$\implies f \in L^\infty((1-\delta_1, 1) \times \mathbf{R}^3)$$

with $\gamma := \frac{5q-6}{2q-2}$.

# Proof of PROP 3: (a) Scaling

•2-parameter group of invariance scaling transfo. for the Landau eq.:

$$f_{\lambda,\epsilon}(t,v) := \lambda f(\lambda t, \epsilon v)$$

•let $f$ be a $(\mathcal{N}, q, C_E)$-suitable solution on $[0,1]$, with $\lambda = \epsilon^\gamma$

$$H_+(f_{\lambda,\epsilon}(t,\cdot)|\epsilon^\gamma\kappa) = \epsilon^{\gamma-3} H_+(f(\epsilon^\gamma t,\cdot)|\epsilon^\gamma\kappa)$$

$$\int_{t_1}^{t_2}\int (f_{\lambda,\epsilon}(t,v) - \epsilon^\gamma\kappa)_+ dvdt = \frac{1}{\epsilon^3}\int_{\epsilon^\gamma t_1}^{\epsilon^\gamma t_2}\int f(T,V) - \kappa)_+ dVdT$$

while $\gamma := \frac{5q-6}{2q-2}$ implies that

$$\int_{t_1}^{t_2}\left(\int |\mathbf{1}_{f_{\lambda,\epsilon}\geq\epsilon^\gamma\kappa}\nabla_v f_{\lambda,\epsilon}^{\frac{1}{q}}(t,v)|^q dv\right)^{2/q} dt$$

$$= \epsilon^{\gamma-3}\int_{\epsilon^\gamma t_1}^{\epsilon^\gamma t_2}\left(\int |\mathbf{1}_{f\geq\kappa}\nabla_v f^{\frac{1}{q}}(T,V)|^q dV\right)^{2/q} dT$$

•Set

$$f_n(t,v) := \epsilon_n^\gamma f(1+\epsilon_n^\gamma(t-1), \epsilon_n v) \quad \text{with } \epsilon_n := 2^{-n}$$

$$F_n(t,v) := \mu((f_n(t,v)^{1/q}-1)_+)\,, \quad \int F_n(t,v)dv \le \epsilon_n^{\gamma-3}$$

•Observe first that $f_n$ is a $(\mathcal{N}_n, q, C_E)$-suitable solution of the Landau equation on $[0,1]$ with

$$\mathcal{N}_n := \{t \ge 0 \text{ s.t. } 1+\epsilon_n^\gamma(t-1) \in \mathcal{N}\}$$

**Key point:** the constant $C_E$ is **unchanged** by the scaling

•There exists $N$ large enough so that

$$n \ge N \implies \int_0^1 \left(\int |\nabla_v F_n(t,v)|^q dv\right)^{2/q} dt$$

$$\le 4\epsilon_n^{\gamma-3} \int_{1-\epsilon_n^\gamma}^1 \left(\int |\mathbf{1}_{f \ge \epsilon_n^{-\gamma}} \nabla_V f(T,V)^{1/q}|^q dV\right)^{2/q} dT < 8\eta_1$$

# Proof of PROP 3: (b) Iteration

•Use the Hölder inequality + Sobolev inequality as in the proof of PROP 2, isolating the term $\|\nabla_v F_{n+1}\|_{L^2_t L^q_v} = O(\eta_1)$ shows that

$$X_m := \operatorname*{ess\,sup}_{\frac{1}{2}<t<1} \int F_{N+m}(t,v)^q dv$$

satisfies

$$X_{m+1} < \rho(\max(1,X_m)^\alpha + \max(1,X_{m-1})^\alpha)\,, \quad X_0, X_1 \le M$$

$$\text{with } \alpha := q/3\,, \qquad \rho := D(q)\eta_1^{q/2}\,, \qquad M := 2^{(N+3)(3-\gamma)}$$

•With $\eta_1 \ll 1$ so that $\rho < \frac{1}{2}$, an easy induction shows that

$$X_{2m}, X_{2m+1} \le \max\left(2\rho, (2\rho)^{\frac{1-\alpha^m}{1-\alpha}} M^{\alpha^m}\right) \implies X_{m_0} < 2D(q)\eta_1^{\frac{q}{2}} \ll 1$$

•Hence $f_{N+m_0+3}$ satisfies the assumption in PROP 2, q.e.d.

•By PROP 1, $f_{in}$ launches at least one $(\mathcal{N}, q, C_E)$ suitable solution with a constant $C_E[T, f_{in}, q]$ for each $q \in (1,2)$

•If $\tau \in \mathbf{S}[f,[1,2]]$, apply PROP 3 to $f_\tau(t,v) := f(t+\tau-1,v)$. Then for each $q \in (\frac{4}{3},2)$, there exists $\epsilon(\tau) \in (0,\frac{1}{2})$ s.t.

$$\int_{\tau-\epsilon(\tau)^\gamma}^{\tau} \left( \int |\nabla_v (f(t,v)^{1/q}-1)_+|^q dv \right)^{2/q} dt \geq \tfrac{1}{2}\eta_1 \epsilon(\tau)^{3-\gamma}$$

•By Vitali's covering thm, there is a sequence $\tau_j \in \mathbf{S}[f,[1,2]]$ s.t.

$$\begin{cases} \mathbf{S}[f,[1,2]] \subset \bigcup_{j\geq 1} (\tau_j - 5\epsilon(\tau_j)^\gamma, \tau + 5\epsilon(\tau_j)^\gamma) \\ \text{and } (\tau_j - \epsilon(\tau_j)^\gamma, \tau + \epsilon(\tau_j)^\gamma) \text{ pairwise disjoint} \end{cases}$$

•Then

$$\frac{1}{2}\eta_1\sum_{j\ge1}\epsilon(\tau_j)^{3-\gamma}\le\sum_{j\ge1}\int_{\tau_j-\epsilon(\tau_j)^\gamma}^{\tau_j}\left(\int|\nabla_v(f(t,v)^{1/q}-1)_+|^q dv\right)^{\frac{2}{q}}dt$$
$$\le\int_0^2\left(\int|\nabla_v(f(t,v)^{1/q}-1)_+|^q dv\right)^{\frac{2}{q}}dt<\infty$$

•Since $\gamma=\frac{5q-6}{2q-2}$, one has $\frac{3-\gamma}{\gamma}=\frac{q}{5q-6}$, and the inequality above proves that

$$\mathcal{H}^{\frac{q}{5q-6}}(\mathbf{S}[f,[1,2]])<\infty\quad\text{for each } q\in(\tfrac{4}{3},2)$$

# Final Remarks/Perspectives

•Analogy between Leray solutions of 3d Navier-Stokes and H-solutions of the Landau equation

| | | |
|---|---|---|
| Navier-Stokes | $u(t,x) \in L^\infty_t L^2(dx)\,,$ | $\nabla_x u \in L^2_t L^2(dx)$ |
| Landau | $\sqrt{f(t,v)} \in L^\infty_t(L^1_2(dv))\,,$ | $\nabla_v \sqrt{f} \in L^2_t(L^2_{-3}(dv)$ |

•As in the case of Leray solutions of the Navier-Stokes equations, H-solutions of the Landau equation become regular after some finite time (Desvillettes-He-Jiang 2022)

•Global existence of a classical solution is unknown not only in the Coulomb case, but also in the (maybe easier?) case where

$$a(z) = \Pi(z)/|z|^\gamma\,, \qquad \text{with } \gamma \in (0,1)$$