Subordinated Markov processes: estimates for heat kernels (and Green functions)

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Joint work with B. Trojan

Cortona, 2022

Introduction

- (M, d) – metric space
- $(M, \mu, \mathcal{B})$ - measure space; $\mathcal{B}$– Borel; μ – Radon
- $\mathcal{T} = (0, \infty)$ or $\mathcal{T} = \mathbb{N}$ – time
- $\{P_t\}_{t \in \mathcal{T} \cup \{0\}}$ – Markov semigroup on $L^2(M, \mu)$.
- $\{\nu_t\}_{t \in \mathcal{T} \cup \{0\}}$ – convolution semigroup of probability measure on $\mathcal{T} \cup \{0\}$.

Subrodination

Define subordinated semigroup

$$\tilde{P}_t f = \int P_s f \nu_t(ds)$$

- $\mathcal{T} = (0, \infty)$
  We have
  $$\mathcal{L}\mu_t(\lambda) = e^{-t\varphi(\lambda)}, \quad \lambda \geq 0,$$
  where
  $$\varphi(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda s}) N(ds) \quad (\mathit{Bernstein\ function})$$
  for some $b \geq 0$ and the measure N.
- $\mathcal{T} = \mathbb{N}$
  $$\nu_1(\{n\}) = a_n, \qquad n \in \mathbb{N}_0$$

Generators

- $\mathcal{T} = (0, \infty)$

  $(\mathcal{A}, D(\mathcal{A}))$ – generator of P_t

  $-\varphi(-\mathcal{A})$ – generator of $\tilde{P}_t$

- $\mathcal{T} = \mathbb{N}$

  $\mathcal{A} = P_1 - I_d$ – "generator" of P_t

  $-f(-\mathcal{A})$ – when "generator" of $\tilde{P}_t$?

Generators

Bendikov, Saloff-Coste (2012):

Let φ be a Bernstein function such that $\varphi(0) = 0$ and $\varphi(1) = 1$.

Then, for

$$\nu_1(\{n\}) = \frac{|\varphi^{(n)}(1)|}{n!}$$

the "generator" of $\tilde{P}_t$ is

$$-\varphi(-\mathcal{A})$$

And

$$\mathcal{L}\nu_t(\lambda) = \left(1 - \varphi\left(1 - e^{-\lambda}\right)\right)^t$$

Examples

Let $s, \beta \in (0, 1]$

- $\varphi(\lambda) = \lambda^s \longrightarrow \quad -(-\mathcal{A})^s$
- $\varphi(\lambda) = (\lambda + 1)^s - 1 \longrightarrow \quad -(I - \mathcal{A})^s + I$
- $\varphi(\lambda) = \lambda^s + \lambda^\beta \longrightarrow \quad -(-\mathcal{A})^s - (-\mathcal{A})^\beta$
- $\varphi(\lambda) = \ln^\beta(1 + \lambda^s) \longrightarrow \quad -\ln^\beta(I - (-\mathcal{A})^s)$

Examples

If $\mathcal{T} = \mathbb{N}$ and $\varphi(\lambda) = \lambda^s$, $s \in (0,1)$ we have

$$\nu_1(\{n\}) = \frac{s\Gamma(n-s)}{\Gamma(1-s)n!} \sim \frac{s}{\Gamma(1-s)} \frac{1}{n^{1+s}}$$

For $\varphi(\lambda) = [(\lambda+1)^s - 1]/(2^s - 1)$

$$\nu_1(\{n\}) = \frac{s\Gamma(n-s)}{\Gamma(1-s)n!} \frac{2^{-n}}{1-2^{-s}} \sim \frac{s}{(1-2^{-s})\Gamma(1-s)} \frac{1}{n^{1+s}2^n}$$

For $\varphi(\lambda) = c\int_0^\infty (1-e^{-\lambda u}) \frac{du}{u\ln^2(2+u)} \approx \ln^{-1}(1+\lambda^{-1})$, $\lambda < 1$

$$\nu_1(\{n\}) \approx \frac{1}{n\ln^2 n}$$

Heat kernel

Assume that

$$P_t f(x) = \int p_t(x,y) f(y) \mu(dy)$$

Then

$$\tilde{P}_t f(x) = \int p_t^{\varphi}(x,y) f(y) \mu(dy),$$

where

$$p_t^{\varphi}(x,y) = \int p_u(x,y) \nu_t(du)$$

(If $\mathcal{T} = (0,\infty)$ we assume that φ is unbounded)

Upper bounds

Proposition

Let $x_0, y_0 \in M$. Assume that there exist $C, c \geq 0$ such that

$$p_u(x_0, y_0) \leq C \min\{1, cu\}, \quad u \in \mathcal{T}.$$

Then

$$p_t^{\varphi}(x_0, y_0) \leq 2C \min\{1, t\varphi(c)\}, \quad t \in \mathcal{T}.$$

Upper bounds

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Let $x_0, y_0 \in M$. Assume that there exist $C, c \geq 0$ such that

$$p_u(x_0, y_0) \leq C \min\{1, cs\}, \quad s \in \mathcal{T}.$$

Then

$$p_t^{\varphi}(x_0, y_0) \leq 2C \min\{1, t\varphi(c)\}, \quad t \in \mathcal{T}.$$

Proof.

$$\begin{aligned} p_t^{\varphi}(x_0, y_0) &\leq C \int \min\{1, cu\}\nu_t(du) \leq 2C \int (1 - e^{-cu})\nu_t(du) \\ &= 2C\,(1 - \mathcal{L}\nu_t(c)) \leq 2Ct\varphi(c). \end{aligned}$$

Upper bounds

Proposition

Fix $x_0, y_0 \in M$, and assume that there exists a non-decreasing function $W : [\inf \mathcal{T}, \infty) \to [0, \infty)$ such that $W(u)u^{-\delta}$ decrease for some $\delta > 0$, and

$$p_u(x_0, y_0) \leq \frac{1}{W(u)} \qquad \text{for all } u \in \mathcal{T}.$$

Suppose that $\varphi(\lambda)/\lambda^\beta$ is nondecreasing on $(0, 1/\inf \mathcal{T})$ for some $\beta \in (0, 1]$ Then there is $C = C(\beta, \delta) > 0$, such that

$$p_t^\varphi(x_0, y_0) \leq \frac{C}{W\left(\frac{1}{\varphi^{-1}(t^{-1})}\right)} \qquad \text{for all } t \in \mathcal{T}.$$

Proof.

Let $d = \lceil 2\delta \rceil$

$$
\begin{aligned}
W\left(\frac{1}{\varphi^{-1}(t^{-1})}\right) p_u(x_0, y_0) &\le \frac{W\left(\frac{1}{\varphi^{-1}(t^{-1})}\right)}{W(u)} \frac{(\varphi^{-1}(t^{-1}))^{\delta}}{u^{-\delta}\left(u\varphi^{-1}(t^{-1})\right)^{\delta}} \\
&\le \max\left\{1, \left(\frac{1}{u\varphi^{-1}(t^{-1})}\right)^{\delta}\right\} \\
&\le 1 + \left(\frac{1}{u\varphi^{-1}(t^{-1})}\right)^{d/2} \\
&= 1 + (\varphi^{-1}(t^{-1}))^{-d/2} c\, p_u^{G-W}(0,0).
\end{aligned}
$$

Upper bounds

Theorem
Fix $x_0, y_0 \in M$, and assume that there exists a non-decreasing function $W : [\inf \mathcal{T}, \infty) \to [0, \infty)$ such that $W(u)u^{-\delta}$ decrease for some $\delta > 0$, and

$$p_u(x_0, y_0) \le \min\left\{\frac{1}{W(u)}, u\frac{c}{W(1/c)}\right\} \qquad \text{for all } u \in \mathcal{T}.$$

Suppose that $\varphi(\lambda)/\lambda^\beta$ is nondecreasing on $(0, 1/\inf \mathcal{T})$ for some $\beta \in (0, 1]$ Then there is $C = C(\beta, \delta) > 0$, such that

$$p_t^\varphi(x_0, y_0) \le C \min\left\{\frac{1}{W\left(\frac{1}{\varphi^{-1}(t^{-1})}\right)}, t\frac{\varphi(c)}{W(1/c)}\right\} \qquad \text{for all } t \in \mathcal{T}.$$

Lower bounds

Theorem

Fix $x_0 \in M$, and assume that there are f and g such that .....

$$P_u[1_{B^c(x_0,r)}](x_0) \le C_1 s f(r) \qquad \text{for all } r > 0,\ s \in \mathcal{T}.$$

and $p_u(x_0, y) \ge g(u, d(x_0, y)) \qquad$ for all $y \in M$, $u \in \mathcal{T}$, and

$$\int_{B(x_0,r)^c} g(u, d(x_0, y))\, \mu(\mathrm{d}y) \ge c_1 > 0 \qquad \text{whenever } u\, f(r) \ge 1.$$

If $\varphi(\lambda)/\lambda^\beta$ is nondecreasing and $\varphi(\lambda)/\lambda^\alpha$ nonincreasing on $(0, 1/\inf \mathcal{T})$ for some $0 < \beta \le \alpha < 1$, then there is $c, \lambda > 0$

$$p_t^\varphi(x_0, y) \ge c \min\left\{ \frac{1}{\mu(B(x_0, \lambda f^{-1}(\varphi^{-1}(1/t))))}, \frac{t\, \varphi(f(d(x_0, y)))}{\mu(B(x_0, \lambda d(x_0, y)))} \right\}$$

Theorem

Suppose that (....) and there exists $C_1 \geq 1$ such that for all $x, y \in M$ and $u > 0$,

$$C_1^{-1} \frac{\mathbf{1}_{\{f(d(x,y)) \leq t\}}}{\mu(B(x, f^{-1}(t)))} \leq p_u(x, y) \leq C_1 \frac{1}{\mu(B(x, f^{-1}(t)))} e^{-g\left(\frac{f(d(x,y))}{t}\right)}$$

The following are equivalent:

1. $\varphi(\lambda)/\lambda^{\beta}$ is nondecreasing and $\varphi(\lambda)/\lambda^{\alpha}$ nonincreasing on $(0, 1/\inf \mathcal{T})$ for some $0 < \beta \leq \alpha < 1$
2. For $x, y \in M$ and $t > 0$

$$p_t^{\varphi}(x, y) \approx \min\left\{ \frac{1}{\mu\left(B\left(x, f^{-1}\left(\frac{1}{\varphi^{-1}(1/t)}\right)\right)\right)}, t\frac{\varphi\left(\frac{1}{f(d(x,y))}\right)}{\mu(B(x, d(x, y)))} \right\}$$

3. For $x, y \in M$

$$\int p_u(x, y) N(du) \approx \frac{\varphi\left(\frac{1}{f(d(x,y))}\right)}{\mu(B(x, d(x, y)))}$$

Theorem

Suppose that (....) and there exists $C_1 \geq 1$ such that for all $x, y \in M$ and $n \in \mathbb{N}$,

$$C_1^{-1} \frac{\mathbf{1}_{\{f(d(x,y)) \leq n\}}}{\mu(B(x, f^{-1}(n)))} \leq p_n(x,y) \leq C_1 \frac{1}{\mu(B(x, f^{-1}(n)))} e^{-g\left(\frac{f(d(x,y))}{n}\right)}$$

The following are equivalent:

1. $\varphi(\lambda)/\lambda^{\beta}$ is nondecreasing and $\varphi(\lambda)/\lambda^{\alpha}$ nonincreasing on $(0, 1/\inf \mathcal{T})$ for some $0 < \beta \leq \alpha < 1$
2. For $x, y \in M$ and $n \in \mathbb{N}$

$$p_n^{\varphi}(x,y) \approx \min \left\{ \frac{1}{\mu\left(B\left(x, f^{-1}\left(\frac{1}{\varphi^{-1}(1/n)}\right)\right)\right)}, n \frac{\varphi\left(\frac{1}{f(d(x,y))}\right)}{\mu(B(x, d(x,y)))} \right\}$$

3. For $x, y \in M$

$$p_1^{\varphi}(x,y) \approx \min \left\{ \frac{1}{\mu\left(B(x, f^{-1}(1))\right)}, \frac{\varphi\left(\frac{1}{f(d(x,y))}\right)}{\mu(B(x, d(x,y)))} \right\}$$

Theorem

Let (G, E) be an infinite connected locally finite weighted graph which is Ahlfors γ-regular, $\gamma \geq 1$. Suppose that

$$C^{-1}\frac{\mathbf{1}_{\{d(x,y)^{\beta} \leq n\}}}{n^{\gamma/\beta}} \leq p_n(x,y) + p_{n+1}(x,y) \leq \frac{C}{n^{\gamma/\beta}} e^{-c\left(\frac{d(x,y)^{\beta}}{n}\right)^{\frac{1}{\beta-1}}}$$

and φ as above. Then

$$p_n^{\varphi}(x,y) \approx \min\left\{\left(\varphi^{-1}(n^{-1})\right)^{\gamma/\beta}, n\frac{\varphi\left(d(x,y)^{-\beta}\right)}{d(x,y)^{\gamma}}\right\}$$

Furthermore, if $\gamma > \beta$, then

$$G_{\varphi}(x,y) \approx \frac{1}{d(x,y)^{\gamma}\varphi(d(x,y)^{-\beta})}$$

Example

Let us consider d-dimensional integer lattice $\mathbb{Z}^d$ with the counting measure. Let p be a finite range *periodic* random walk on $\mathbb{Z}^d$, for example simple random walk.

$$p_n^{\varphi}(x,y) \approx \min \left\{ \left(\varphi^{-1}(n^{-1})\right)^{d/2}, n\frac{\varphi\left(d(x,y)^{-2}\right)}{d(x,y)^d} \right\}$$

Cygan and Šebek (2021) for complete Bernstein functions.

Stability

Theorem
(M, d, μ) – discrete metric measure space that satisfies (...) and ψ – increasing function satisfying (...). Suppose that there exists a semigroup such that

$$p_n(x,y) \approx \min\left\{\frac{1}{\mu(B(x,\psi^{-1}(n)))}, \frac{n}{\mu(B(x,d(x,y)))\psi(d(x,y))}\right\}$$

If $Qf(x) = \int q_1(x,y)f(y)\mu(dy)$ and $q_1 \approx p_1$ then, for $Q^n \sim q_n$,

$$q_n(x,y) \approx p_n(x,y), \quad n \in \mathbb{N},\ x,y \in M.$$

Discrete version of the result of Chen, Kumagai, Wang (2021).

References

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Thank you!