Lagrangian stochastic models for turbulent flows and related problems

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Lagrangian stochastic models for turbulent flows : A class of stochastic processes aimed to describe and simulate the evolution of a generic fluid particle issued from a turbulent flow.

Generic model : Langevin's type dynamic of the form :

$$\begin{cases} X_t = X_0 + \int_0^t U_s \, ds, \\ U_t = U_0 + \int_0^t B(s, X_s, U_s, \mathbb{E}[U_s \,|\, X_s]) \, ds + \int_0^t \sigma(s, X_s, \mathbb{E}[U_s \,|\, X_s]) \, dW_s \\ \qquad + \text{Physical constraints}. \end{cases}$$

where $(W_t;\ t \geq 0)$ is a $\mathbb{R}^d$-standard Brownian motion and for B and σ are given functions.

Framework : Computational fluid dynamic and the statistical description of turbulent flows.

Reynolds decomposition's principle : All the macroscopic properties of a fluid flow, e.g. velocity U and pressure P, can be decomposed into a averaged (deterministic) part and a turbulent (random) part :

$$U(t,x,w) = \langle U\rangle(t,x) + u(t,x,w),\ \ P(t,x,w) = \langle P\rangle(t,x) + p(t,x,w),\ \ldots$$

Reynolds-averaged Navier-Stokes (RANS) equations : For an incompressible Newtonian turbulent flow with constant mass ϱ and kinematic viscosity ν, $\langle U\rangle$ and $\langle P\rangle$ are governed by the Reynolds equations :

$$\begin{aligned}
&\partial_t\langle U\rangle + (\langle U\rangle\cdot\nabla_x)\langle U\rangle = -\frac{1}{\varrho}\nabla_x\langle P\rangle - \nabla_x\cdot\langle u\otimes u\rangle + \nu\triangle_x\langle U\rangle,\\
&(\nabla_x\cdot\langle U\rangle) = 0,
\end{aligned}$$

where

$$\langle u\otimes u\rangle = \langle U\otimes U\rangle - \langle U\rangle\otimes\langle U\rangle,$$

is the so-called Reynolds (stress) tensor.

Turbulence models $\Rightarrow$ Closure of the equations with a choice of a model for the tensor $\langle u\otimes u\rangle$ (Sagaut 1986, Mohammadi and Pironneau 1994, Schiestel 1998, Pope 2000, ...).

Lagrangian interpretation :

Law of $X_t \leftrightarrow \overline{\rho}(t,x)$, mass distibution of the fluid flow,
$\mathbb{E}[U_t \mid X_t = x] \leftrightarrow \langle U\rangle(t,x)$, bulk velocity at time t and point x,
$\mathbb{E}\left[(U_t - \mathbb{E}[U_t \,|\, X_t]) \otimes (U_t - \mathbb{E}[U_t \,|\, X_t]) \mid X_t = x\right] \leftrightarrow$ Reynolds tensor.

Equations of motion : Fokker-Planck equation related to (X_t, U_t) :

$$\begin{aligned}0 &= \partial_t \rho(t,x,u) + (u \cdot \nabla_x \rho(t,x,u)),\\ &+ \nabla_u \cdot (B(t,x,\langle U\rangle(t,x))\rho(t,x,u)) - \frac{1}{2}\mathrm{Trace}\left(\nabla_u^2 \otimes (\sigma\sigma^*(t,x,\langle U\rangle(t,x)\rho(t,x,u)\right).\end{aligned}$$

- Integrating on the velocity space $\Rightarrow$ continuity equation (in compressible form) :

$$\partial_t \overline{\rho}(t,x) + \nabla_x \cdot (\overline{\rho}(t,x)\langle U\rangle(t,x)) = 0.$$

- Multiplying by u_i and integrating over velocity space $\Rightarrow$

$$\partial_t \int u_i \rho(t,x,u)du + \sum_{j=1}^{d} \int u_i u_j \partial_{x_j} \rho(t,x,u)du = -\int B^{(i)}(t,x,\langle U\rangle(t,x))\rho(t,x,u)du$$

$$\Leftrightarrow \; \partial_t(\overline{\rho}\langle U^{(i)}\rangle)(t,x) + \sum_{j=1}^{d} \partial_{x_j}(\overline{\rho}\langle U^{(i)}U^{(j)}\rangle)(t,x) = -\int B^{(i)}(t,x,\langle U\rangle(t,x))\rho(t,x,u)du.$$

Characteristic of the approach :
◦ Simulation of complex turbulent flows with Monte–Carlo approximation, systems of stochastic particles ;

◦ Different approach than classical and more direct probabilistic interpertation of Navier-Stokes or Euler equations e.g. Chorin 1973-1978 ; Marchioro and Pulvirenti 1982 ; Constantin and Iyer 2008.

Applications :

- Wall bounded flows (Dreeben and Pope 1997) ;

- Turbulent-reactive flows (i.e. presence of chemical reactions) (Pope 1985 ; Minier and Peirano 2001) ;

- Filtering of meteorological data (Baehr 2008) ;

- Downscaling stochastic methods for wind speed prediction and wind farm production (2004- 2016) : Collaboration between INRIA (France & Chile), ADEME, EDF and LMD (Bernardin *et al.* 2010 ; Bossy *et al.* 2016 ; WindPos project ; ...)

A prototypical Lagrangian stochastic model for a incompressible turbulent flow (Pope (1994)) :

$$dX_t = U_t dt,$$
$$dU_t = \left(-\frac{1}{\varrho}\nabla_x P(t,X_t) + G(t,X_t)\left(\mathbb{E}[U_t \mid X_t] - U_t\right)\right) dt + C(t,X_t)dW_t,$$

where the pressure gradient $\nabla_x P(t,x)$ is assumed to satisfy the equation :

$$-\nabla_x \cdot \left(\overline{\rho}(t,x)\nabla P(t,x)\right) = \sum_{i,j} \partial^2_{x_i x_j}\left(\overline{\rho}(t,x)\mathbb{E}\left[U_t^{(i)} U_t^{(j)} \mid X_t = x\right]\right).$$

In particular the pressure gradient must ensure the uniform repartition of the fluid mass and the mean incompressibility constraint

$$\nabla_x \cdot \langle U\rangle(t,x) = 0, \ \forall\, t,x.$$

Characteristic questions :

- Lagrangian models are McKean-Vlasov models type singular nonlinearities (conditional form) and diffusion coefficient partially degenerated $\Rightarrow$ Wellposedness of a solution ? Simplified model : Bossy, J. and Talay 2011.
- Modeling of boundary conditions ? Specular boundary condition : Bossy and J. 2011, 2015, 2018.
- Incompressibility constraint $\Leftrightarrow$ Highly singular coupling of nonlinear SDE–PDE $\Rightarrow$ Wellposedness ? Simplified model : Bossy, Fontbona, Jabin and J. 2013.

Simplified Model of Lagrangian dynamic :

$$(1) \quad \begin{cases} X_t = X_0 + \int_0^t U_s \, ds, \\ U_t = U_0 + \int_0^t \mathbb{E}\left[b(U_s) \,|\, X_s\right] \, ds + \sigma W_t, \end{cases}$$

where $\sigma > 0$, $(X_0, U_0) \sim \mu_0$ and $b : \mathbb{R}^d \to \mathbb{R}^d$ Borel bounded.
Related Fokker-Planck equation :

$$\begin{cases} \partial_t \rho_t(x,u) + u \cdot \nabla_x \rho + \nabla_u \cdot (\rho_t(x,u) B[x; \rho_t]) - \frac{\sigma^2}{2} \triangle_u \rho_t(x,u) = 0 \text{ in } (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d, \\ \rho(t=0,x,u) = \rho_0(x,u) \text{ on } \mathbb{R}^d \times \mathbb{R}^d. \end{cases}$$

for

$$B[x; \rho_t] = \begin{cases} \dfrac{\int b(v) \rho_t(x,v) \, dv}{\int \rho_t(x,v) \, dv} \text{ if } \int \rho_t(x,v) \, dv \neq 0, \\ 0 \text{ otherwise.} \end{cases}$$

Mild/Duhamel formulation : For S_t the semi-group of $(x + \int_0^t V_s \, ds, v + \sigma W_t)$,

$$\rho_t(x,u) = S_t(\mu_0)(x,u) + \int_0^t \nabla_v S_{t-s}(\rho_s B[\cdot; \rho_s])(x,u) \, ds$$

Note :

$$|\int \varrho(x,v) \, dv B[x; \varrho] - \int \tilde{\varrho}(x,v) \, dv B[x; \tilde{\varrho}]| \leq C \int |\varrho(x,v) - \tilde{\varrho}(x,v)| \, dv$$

$\Rightarrow$ Uniqueness of the mild equation holds in $L^\infty([0,T]; L^1(\mathbb{R}^d \times \mathbb{R}^d))$ for $T < \infty$ arbitrary.

• Regularization of the continuous interaction kernel :

$$\begin{cases} X_t^\epsilon = X_0 + \int_0^t U_s^\epsilon \, ds, \\ U_t^\epsilon = U_0 + \int_0^t B_\epsilon[X_s^\epsilon; \rho^\epsilon(s)] \, ds + \sigma W_t, \end{cases}$$

where $\{\phi_\epsilon\}_{\epsilon>0}$ is a family of non-negative smooth probability density function approximated the Dirac measure and

$$B_\epsilon\left[x; \rho^\epsilon(t)\right] := \frac{\mathbb{E}\left[b(U_t^\epsilon)\phi_\epsilon(X_t^\epsilon - x)\right]}{\mathbb{E}\left[\phi_\epsilon(X_t^\epsilon - x)\right] + \epsilon}.$$

For any $\epsilon > 0$, the SDE is wellposed.

Theorem

As $\epsilon \to 0$, $(X_t^\epsilon, U_t^\epsilon;\ t \in [0,T])$ converges weakly to the unique solution of the equation in (X_t, U_t). Moreover, for all $f \in \mathcal{C}_b(\mathbb{R}^{2d})$,

$$\forall\, t \in [0,T], \lim_{\epsilon \to 0^+} \rho^\epsilon(t) = \rho(t), \text{ in } L^1(\mathbb{R}^{2d}).$$

Generic model with physical constraint : Given $\mathcal{D} \subset \mathbb{R}^d$ closed,

$$\begin{cases} X_t = X_0 + \int_0^t U_s \, ds, \\ U_t = U_0 + \int_0^t \mathbb{E}[b(U_s) \,|\, X_s] \, ds + W_t + K_t, \end{cases}$$

where $(K_t;\ t \geq 0)$ imposes that $X_t \in \mathcal{D}$, for all t, and models the possible interaction between the "particle" and the wall $\partial\mathcal{D}$.

Kinetic Fokker-Planck equation with boundary condition : Given $n_{\mathcal{D}}$ the unit outward normal vector of $\mathcal{D}$,

$$\Sigma^+ = \left\{(x, u) \in \partial\mathcal{D} \times \mathbb{R}^d \,|\, (u \cdot n_{\mathcal{D}}(x)) > 0\right\} \text{ ("outgoing" particle state space)},$$

$$\Sigma^- = \left\{(x, u) \in \partial\mathcal{D} \times \mathbb{R}^d \,|\, (u \cdot n_{\mathcal{D}}(x)) < 0\right\} \text{ ("emerging" particle state space)},$$

and $\gamma^{\pm}(\rho)$ the trace of ρ on $\Sigma^{\pm}$ respectively,

$$\begin{cases} \partial_t \rho + u \cdot \nabla_x \rho + \nabla_u \cdot (\rho B[\cdot; \rho]) - \frac{\sigma^2}{2} \triangle_u \rho = 0 \text{ on } (0, \infty) \times \mathcal{D} \times \mathbb{R}^d, \\ \rho(t = 0, x, u) = \rho_0(x, u) \text{ on } \mathcal{D} \times \mathbb{R}^d, \\ \gamma^-(\rho)(t, x, u) = \Gamma(t, x, u; \gamma^+(\rho)) \text{ on } (0, \infty) \times \partial\mathcal{D} \times \mathbb{R}^d. \end{cases}$$

Maxwell boundary condition (Cercignani *et al.* '94) :

- complete reflection :

$$\gamma^{-}(\rho)(t,x,u) = \gamma^{+}(\rho)(t,x,-u),\ (x,u) \in \Sigma^{-}.$$

- specular boundary condition (elastic wall) :

$$\gamma^{-}(\rho)(t,x,u) = \gamma^{+}(\rho)(t,x,u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)),\ (x,u) \in \Sigma^{-}.$$

- diffusive :

$$\gamma^{-}(\rho)(t,x,u) = M_{\Theta}(u) \int_{v \cdot n_{\mathcal{D}}(x) > 0} \gamma^{+}(\rho)(t,x,v)\, dv,\ (x,u) \in \Sigma^{-},$$

where M_{Θ} is a Maxwellian distribution of the form :

$$M_{\Theta}(u) = \frac{1}{(2\pi)^{\frac{d-1}{2}} \Theta^{\frac{d+1}{2}}} e^{-\frac{|u|^2}{2\Theta}},\ u \cdot n_{\mathcal{D}}(x) < 0$$

Trace problem ($\Leftrightarrow$ Give a meaning to the trace functions $\gamma^{\pm}(\rho)$) : Degond and Mas-Gallic 1987, Carrillo 1998, Mischler 2010, Nier 2015, ...

- In the case $\sigma = 0$ and partially absorbing boundary : second order ODEs with impact : Schatzman 1998, Ballard 2000.
- General diffusive-reflective boundary condition in the case $W \to$ Poisson process : Costantini 1991, Costantini and Kurtz 2006.
- Case of an absorbing wall at the frontier of $\mathcal{D} = (0, \infty)$: Bertoin 2007, 2008 : Existence and uniqueness of a solution to

$$\begin{cases} X_t = X_0 + \int_0^t U_s \, ds, \\ U_t = U_0 + B_t - \sum_{0 < s \leq t} U_{s-} \mathbb{1}_{\{X_s = 0\}}, \end{cases}$$

  for $(B_t;\ t \geq 0)$ a one-dimensional standard Brownian motion.
- Jacob 2012, 2013 : Case of a partially absorbing wall :

$$\begin{cases} X_t = X_0 + \int_0^t U_s \, ds, \\ U_t = U_0 + B_t - (1 + c) \sum_{0 < s \leq t} U_{s-} \mathbb{1}_{\{X_s = 0\}}, \ 0 \leq c \leq 1. \end{cases}$$

- J. and Profeta 2018 : Case of a Langevin dynamic driven by an α-stable ($\alpha \in (0, 2]$) Levy process and endowing mixed diffusive-reflective boundary condition.

Lagrangian stochastic model with specular boundary condition :

$$\begin{cases} X_t = X_0 + \int_0^t U_s \, ds, \\ U_t = U_0 + \int_0^t \mathbb{E}[b(U_s) \,|\, X_s] \, ds + W_t + K_t, \\ K_t = -2 \sum_{0<s\leq t} (U_{s^-} \cdot n_{\mathcal{D}}(X_s)) \, n_{\mathcal{D}}(X_s) \mathbb{1}_{\{X_s \in \partial\mathcal{D}\}}. \end{cases}$$

In this case, if $X_t \in \partial\mathcal{D}$,

$$U_{t^+} = U_{t^-} - 2 \, (U_{t^-} \cdot n_{\mathcal{D}}(X_t)) \, n_{\mathcal{D}}(X_t).$$

Note : Specular boundary condition $\Rightarrow$ Mean no-permeability condition :

$$x \in \partial\mathcal{D}, \;\; \mathbb{E}[(U_t \cdot n_{\mathcal{D}}(X_t)) \,|\, X_t = x] = 0 \Leftrightarrow (\langle U \rangle(t,x) \cdot n_{\mathcal{D}}(x)) = 0.$$

Core problem : Existence and growth to infinity of the sequence of hitting times :

$$\tau_n = \inf \{ t > \tau_{n-1} \,|\, X_t \in \partial\mathcal{D} \} \, , \;\; \tau_0 = 0,$$

is needed to show that the confinement component

$$K_t = -2 \sum_{n \in \mathbb{N}} \left(U_{\tau_n^-} \cdot n_{\mathcal{D}}(X_{\tau_n}) \right) n_{\mathcal{D}}(X_{\tau_n}) \mathbb{1}_{\{\tau_n \leq t\}}$$

is well defined at all time.

$$(2) \quad \begin{cases} X_t = X_0 + \int_0^t U_s \, ds, \\ U_t = U_0 + W_t + \int_0^t \mathbb{E}[b(U_s) \,|\, X_s] \, ds - 2 \sum_{0 < s \leq t} U_{s-} \mathbb{1}_{\{X_s = 0\}}, \end{cases}$$

for $(W_t;\ t \geq 0)$ a standard (one-dimensional) Brownian motion.

◦ **Preliminary results (case $b = 0$) :** Consider the "free" Langevin dynamic :

$$\begin{cases} Y_t^{x,u} = x + \int_0^t V_s^{x,u} \, ds, \\ V_t^{x,u} = u + W_t, \end{cases}$$

for $x, u \in \mathbb{R}$. Then

- McKean 1963 established that : If $(x, u) \neq (0, 0)$ then, a.s., all path $t \mapsto (Y_t^{x,u}, V_t^{x,u})$ never hit $(0, 0)$.

- Lachal 1997 derived the explicit expression for the joint law of $(\beta_n^{x,u}, V_{\beta_n}^{x,u})$ where

$$\beta_{n+1}^{x,u} = \inf \left\{ t > \beta_n^{x,u} \,|\, Y_t^{x,u} = 0 \right\}.$$

Proposition (Linear case)

Assuming that $supp(\mu_0) \subset (0, \infty) \times \mathbb{R}$, *the process*

$$X_t = |Y_t|,\ U_t = sign(Y)_{t^+} V_t,$$

for

$$(sign(Y)_{t^+};\ t \in [0, T]),\ \text{the càdlàg modification of } sign(Y_t)$$

is the unique (in the strong sense) solution the SDE (2) *for* $b = 0$.

Proposition (Non-linear case)

Assume also that $\mathbb{E}[|X_0| + |U_0|^2] < \infty$, $\mu_0(dx, du) = \rho_0(x, u)\, dxdu$, *and* $b : \mathbb{R} \to \mathbb{R}$ *is bounded. Then there exists a unique solution (in the weak sense) to*

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t U_s\, ds, \\ U_t = U_0 + \displaystyle\int_0^t \mathbb{E}[b(U_s) \,|\, X_s]\, ds + B_t - 2 \sum_{0<s\leq t} U_{s-} \mathbb{1}_{\{X_s = 0\}}. \end{cases}$$

In addition, for all $t \in [0, T]$, $\mathcal{L}(X_t, U_t)$ *admits a probability density function* $\rho(t, x, u)$ *such that, for a.a.* $(t, u) \in (0, T) \times \mathbb{R}$, $x \mapsto \rho(t, x, u)$ *is continuous on* $[0, \infty)$ *and*

$$\rho(t, 0, u) = \rho(t, 0, -u).$$

$$(3)\quad \begin{cases} X_t = X_0 + \int_0^t U_s\, ds, \\ U_t = U_0 + \int_0^t \mathbb{E}\left[b(U_s)\,|\,X_s\right]\, ds + W_t + K_t, \\ K_t = -2 \sum_{0\leq s\leq t} (U_{s^-}\cdot n_{\mathcal{D}}(X_s))\, n_{\mathcal{D}}(X_s)\, \mathbb{1}_{\{X_s\in\partial\mathcal{D}\}},\ \forall t\in[0,T]. \end{cases}$$

General assumptions :

(A_1) $\mathrm{supp}(\mu_0) \subset \mathcal{D}\times\mathbb{R}^d$ and $\mu_0(dx,du) = \rho_0(x,u)dxdu$,

(A_2) $\mathcal{D}$ is an open bounded subset of $\mathbb{R}^d$ and its boundary $\partial\mathcal{D}$ is of class $\mathcal{C}^3$.

(A_3) $b:\mathbb{R}^d\to\mathbb{R}$ is a Borel bounded vector field.

(A_4) (Initial Maxwellian bounds) There exist $\underline{P}_0, \overline{P}_0 : \mathbb{R}^d \to (0,\infty)$ such that

$$\underline{P}_0(u) \leq \rho_0(x,u) \leq \overline{P}_0(u),\ (x,u)\in\mathcal{D}\times\mathbb{R}^d,$$

$$\int_{\mathbb{R}^d} \omega(u)\overline{P}_0(u)\, du < \infty,\ \underline{P}_0(u) > 0,$$

for $\omega(u) = (1+|u|^2)^{\frac{\alpha}{2}}$, $\alpha > d+2$.

Case $b = 0$ **:**

Lemma

Under the assumptions (A_1) *and* (A_2)*, there exists a unique (pathwise) solution to the SDE*

$$\begin{cases} Y_t = X_0 + \int_0^t V_s\, ds, \\ V_t = U_0 + W_t + K_t, \\ K_t = -2 \sum_{0 \leq s \leq t} \left(V_{s-} \cdot n_{\mathcal{D}}(Y_s) \right) n_{\mathcal{D}}(Y_s)\, \mathbb{1}_{\{Y_s \in \partial\mathcal{D}\}}, \ \forall t \in [0, T], \end{cases}$$

and the sequence $\{\tau_n;\ n \in \mathbb{N}\}$ *is well defined and increases to* ∞ *as* n *grows to* ∞*.*

Elements of proof : Using a family of local charts $\{\mathcal{O}_i, \psi_i\}_{i=1,\dots,M}$ related to $\mathcal{D}$ and the straightening from $\mathcal{D}$ to $\mathbb{R}^{d-1} \times (0, \infty)$, the problem is reduced to a one dimensional confinement and the excursion of $(X_t, U_t;\ t \in [0, T])$ in $\mathcal{O}_i \cap \overline{\mathcal{D}}$ can be constructed simply. □

Case $b \neq 0$.

Strategy :

Step 1 : Construction of a solution to the nonlinear parabolic equation :

$$
(4)\quad \begin{cases}
\partial_t \rho(t,x,u) + (u \cdot \nabla_x \rho(t,x,u)) + \nabla_u \cdot (\rho(t,x,u) B[x;\rho(t)]) - \frac{1}{2} \triangle_u \rho(t,x,u) = 0, \\
\rho(0,x,u) = \rho_0(x,u) \text{ on } \mathcal{D} \times \mathbb{R}^d,\; B[x;\rho(t)] = \dfrac{\int b(v)\rho(t,x,v)\,dv}{\int \rho(t,x,v)\,dv}, \\
\gamma^-(\rho)(t,x,u) = \gamma^+(\rho)(t,x,u - 2(u \cdot n_{\mathcal{D}}(x)) n_{\mathcal{D}}(x)) \text{ on } (0,T) \times \Sigma^-.
\end{cases}
$$

Step 2 **: Construction of a weak solution to the SDE** : Starting from the solution ρ^{pde} to (4), construct

$$
\begin{cases}
X_t = X_0 + \int_0^t U_s\, ds, \\
U_t = U_0 + \int_0^t B[X_s, \rho^{pde}(s)]\, ds + \sigma W_t + K_t, \\
K_t = \sum_{0<s\leq t} 2(U_{s^-} \cdot n_{\mathcal{D}}(X_s)) n_{\mathcal{D}}(X_s) \mathbb{1}_{\{X_s \in \partial\mathcal{D}\}}.
\end{cases}
$$

Step 3 : Weak uniqueness of the SDE (3) $\Leftarrow$ Uniqueness of the pde (4).

Step 1.

Proposition

Under the assumptions $(A_1), (A_2)$*,* (A_3) *and* (A_4)*, there exists a unique solution* $\rho \in \mathcal{C}([0, T]; L^2(\mathcal{D} \times \mathbb{R}^d))$ *to* (4) *such that*

$$\sqrt{\omega}\rho, \sqrt{\omega}\nabla_u \rho \in L^2(Q_T), \text{ for } \omega(u) = (1 + |u|^2)^s,\ s > 2(d+3).$$

Moreover ρ *admits a trace* $\gamma^{\pm}(\rho)$ *satisfying the specular boundary condition. Finally, we have the following estimates :*

$$\underline{P} \leq \rho \leq \overline{P}, \text{ on } Q_T,$$
$$\underline{P} \leq \gamma^{\pm}(\rho) \leq \overline{P}, \text{ on } \Sigma_T^{\pm},$$

for

$$\overline{P}(t,u) = e^{\overline{a}t}\left(G_t * \overline{P}_0^{\frac{1}{\overline{\mu}}}(u)\right)^{\overline{\mu}}, \ \underline{P}(t,u) = e^{\underline{a}t}\left(G_t * \underline{P}_0^{\frac{1}{\underline{\mu}}}(u)\right)^{\underline{\mu}}$$

where G_t *is the centered Gaussian density function with variance* t*, and where* $\overline{\mu}, \underline{\mu}, \overline{a}, \underline{a}$ *are positive constant depending only on* d*,* T*,* σ *and* $\|b\|_{L^\infty}$*.*

Steps 2 & 3.

Theorem

Assuming $(A_1), (A_2)$, (A_3) *et* (A_4), *there exists a unique (weak) solution to*

$$\begin{cases} X_t = X_0 + \int_0^t U_s \, ds, \\ U_t = U_0 + \sigma W_t + \int_0^t \mathbb{E}[b(U_s) \,|\, X_s] \, ds + K_t, \\ K_t = -2 \sum_{0<s\leq t} (U_{s^-} \cdot n_{\mathcal{D}}(X_s)) \, n_{\mathcal{D}}(X_s) \mathbb{1}_{\{X_s \in \partial\mathcal{D}\}}. \end{cases}$$

such that, for all t, *the law of* (X_t, U_t) *admits as a probability density function* $\rho(t)$ *solution to* (4).

Corollary

For all $t > 0$, $x \in \mathcal{D}$,

$$\mathbb{E}[(U_t \cdot n_{\mathcal{D}}(X_t)) \,|\, X_t = x] = 0$$

Generic : For $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ the flat d-dimensional torus, $[\cdot]$ the related projection operator and given $\beta, \sigma \geq 0$, be fixed constants.

$$\begin{cases} X_t = \left[X_0 + \int_0^t U_s\, ds\right], \\ U_t = U_0 - \int_0^t \nabla_x P(s, X_s)\, ds + \sigma W_t \end{cases}$$

with the constraints

- for all $t \in [0, T]$, $\mathbb{P}(X_t \in dx) = dx$, (Uniform mass repartition),
- for all $t \in [0, T], x \in \mathbb{T}^d \nabla_x \cdot \mathbb{E}[U_t \,|\, X_t = x] = 0$ (Mean divergence free).

Constraints $\Leftrightarrow$ Uniform mass and divergence free satisfied at time $t = 0$ and

$$\nabla_x \cdot (\overline{\rho}(t,x)\nabla_x P(t,x))) = -\sum_{i,j=1}^{d} \partial^2_{x_i,x_j}\Big(\overline{\rho}(t,x)\mathbb{E}[U_t^{(i)} U_t^{(j)} \,|\, X_t = x]\Big),\ \overline{\rho}(t,x) = \int \rho(t,x,v)\, dv.$$

Related Fokker-Planck equation :

$$\partial_t f(t,x,u) + u \cdot \nabla_x f(t,x,u) - \nabla_u f(t,x,u) \cdot \nabla_x P(t,x) - \frac{\sigma^2}{2} \triangle_u f(t,x,u) = 0 \text{ on } \mathbb{T}^d \times \mathbb{R}^d,$$

$$f(0,x,u) = f_0(x,u) \text{ on } \mathbb{T}^d \times \mathbb{R}^d,$$

$$\nabla_x \cdot (\overline{\rho}(t,x)\nabla_x P(t,x))) = -\sum_{i,j=1}^{d} \partial^2_{x_i,x_j}\Big(\int_{\mathbb{R}^d} v_i v_j f(t,x,v) dv\Big).$$

One-dimensional case :

$$(VFP)\begin{cases}\partial_t f(t,x,u)+u\cdot\nabla_x f(t,x,u)-\nabla_u f(t,x,u)\cdot\nabla_x P(t,x)-\dfrac{\sigma^2}{2}\triangle_u f(t,x,u)=0\\ f(0,x,u)=f_0(x,u)\text{ on }\mathbb{T}\times\mathbb{R},\\ \partial_x(\overline{\rho}(t,x)\partial_x P(t,x))=-\partial_x^2\Big(\displaystyle\int_{\mathbb{R}^d}|v|^2 f(t,x,v)dv\Big)\text{ on }[0,T]\times\mathbb{T}.\end{cases}$$

Corollary

Let f be an analytic solution to

$$(VFP')\begin{cases}\partial_t f(t,x,u)+u\cdot\nabla_x f(t,x,u)-\nabla_u f(t,x,u)\cdot\nabla_x P(t,x)-\dfrac{\sigma^2}{2}\triangle_u f(t,x,u)=0\\ f(0,x,u)=f_0(x,u)\ on\ \mathbb{T}\times\mathbb{R},\\ P(t,x)=-\displaystyle\int_{\mathbb{R}}|v|^2 f(t,x,v)dv\ on\ [0,T]\times\mathbb{T},\end{cases}$$

and assume that $\int f_0(x,v)\,dv=1$ *and* $\partial_x\int vf_0(x,v)\,dv=1$. *Then f is also solution to* (VFP).

Solution spaces (initially introduced in Jabin and Nouri 2011) : From the analytic norm :

$$\|\psi\|_\lambda := \sum_{k,l\in\mathbb{N}} \frac{\lambda^{k+l}}{k!l!} \left\|\partial_x^k \partial_u^l \psi\right\|_\infty, \ \Psi : (x,u) \in \mathbb{T}\times\mathbb{R} \to \Psi(x,u) \in \mathbb{R}.$$

and for $\|\psi\|_{\lambda,a} = \frac{d^a \|\psi\|_\lambda}{d^a \lambda}$, define the norm and semi-norms :

$$\|\psi\|_{\mathcal{H},\lambda} := \sum_{a\in\mathbb{N}} \frac{1}{(a!)^2} \|\psi\|_{\lambda,a}, \qquad \|\psi\|_{\tilde{\mathcal{H}},\lambda} := \sum_{a\geq 1} \frac{a^2}{(a!)^2} \|\psi\|_{\lambda,a}.$$

and the related function spaces :

$$\mathcal{H}(\lambda) := \left\{\psi \in \mathcal{C}^\infty(\mathbb{T}\times\mathbb{R}) \,:\, \|\psi\|_{\mathcal{H},\lambda} < +\infty\right\},$$
$$\widetilde{\mathcal{H}}(\lambda) := \left\{\psi \in \mathcal{C}^\infty(\mathbb{T}\times\mathbb{R}) \,:\, \|\psi\|_{\tilde{\mathcal{H}},\lambda} < +\infty\right\}.$$

Advantage : Offer a more precise control on the convergence near the radius of analyticity.

Weighted equation : For $\omega : \mathbb{R} \to \mathbb{R}$ a smooth function, $g(t,x,u) := \omega(u)f(t,x,u)$, $h(u) = \frac{\partial_u^2 \omega(u)}{2\omega(u)} - |\partial_u \ln(\omega(u))|^2$ satisfy

$$(VFP_w)\begin{cases} \partial_t g + u\partial_x g - [\partial_x P - \partial_u \ln \omega]\,\partial_u g - \dfrac{\sigma^2}{2}\partial_u^2 g = g\partial_x P \partial_u \ln \omega - gh \text{ on } (0,T] \times \mathbb{T} \times \mathbb{R}, \\ g(0,x,u) = g_0(x,u) \text{ on } \mathbb{T} \times \mathbb{R} \\ P(t,x) = -\displaystyle\int_{\mathbb{R}} \frac{u^2}{\omega(u)} g(t,x,u)du. \end{cases}$$

Requirement on the weight : $\displaystyle\lim_{|u|\to+\infty} \frac{\omega(u)}{|u|} = +\infty$, $\displaystyle\int_{\mathbb{R}} \frac{u^2}{\omega(u)} du = 1$,

$$\limsup_{|u|\to\infty} \Big|\frac{\omega'(u)}{\omega(u)}\Big| + \limsup_{|u|\to\infty} \Big|\frac{\omega''(u)}{\omega'(u)}\Big| < \infty,$$

Moreover, for some $\lambda_0 > 0$ we have $\ln(\omega) \in \tilde{\mathcal{H}}(\lambda_0)$ and $h \in \mathcal{H}(\lambda_0)$.
Example : For $s \geq 4$, a positive integer $\omega(u) := c(s)(1+u^2)^{\frac{s}{2}}$ for all value $\lambda_0 \in (0, \frac{1}{4})$, where $c(s) > 0$ is such that $\int_{\mathbb{R}} \frac{u^2}{\omega(u)}\, du = 1$.

Given K, T and λ_0 strictly positive real numbers such that $\lambda_0 > T(1+K)$, and the function

$$\lambda(t) := \lambda_0 - (1+K)t,$$

we now define the spaces

$$\mathcal{B}^M_{\lambda_0,K,T} := \left\{ \psi \in \mathcal{H}_{\lambda_0,K,T} \ : \ \sup_{t\in[0,T]} \|\psi(t)\|_{\mathcal{H},\lambda(t)} \leq M \right\},$$

$$\widetilde{\mathcal{B}}^M_{\lambda_0,K,T} := \left\{ \psi \in_{\lambda_0,K,T} : \int_0^T \|\psi(t)\|_{\tilde{\mathcal{H}},\lambda(t)} dt \leq M \right\}.$$

Theorem

For $\gamma_0 := \|\ln(\omega)\|_{\tilde{\mathcal{H}},\lambda_0}$ *and* $\gamma_1 := \|h\|_{\mathcal{H},\lambda_0}$, *assume that all the three following properties holds*

a) $T < \frac{\lambda_0}{2+\lambda_0+4\gamma_0}$;

b) $M \leq \frac{1}{16}(K - \lambda_0 - 4\gamma_0 - 1)$ *for some* K *in* $(1+\lambda_0+4\gamma_0, \frac{\lambda_0}{T} - 1)$;

c) $M(1+\gamma_0)\exp\{(M\gamma_0+\gamma_1)T\} < 1$.

Assume moreover that f_0 *is of class* $\mathcal{C}^\infty$ *and that* $g_0(x,u) := \omega(u) f_0(x,u)$ *satisfies*

d) $\max\{\|g_0\|_{\mathcal{H},\lambda_0}, T\|g_0\|_{,\lambda_0}\} \leq M$;

e) $\|g_0\|_{\mathcal{H},\lambda_0} \exp(T(\gamma_1 + 16\gamma_0)) \leq M \exp(-(16+\gamma_0)M)$.

Then (VFP_w) *has a unique smooth solution* $g \in \mathcal{B}^M_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^M_{\lambda_0,K,T}$.

Corollary

Let T be given as in the previous theorem. Then a classic smooth solution f to (VFP') exists in $[0,T]\times\mathbb{T}\times\mathbb{R}$. This solution further satisfies $\int_{\mathbb{R}} f(t,x,u)du = 1$ and $\partial_x \int_{\mathbb{R}^d} uf(t,x,u)du = 0$ at all $(t,x)\in[0,T]\times\mathbb{T}$.

Corollary

∘ *There exists in $[0,T]$ a solution tothe stochastic differential equation*

$$\begin{cases} X_t = \left[X_0 + \int_0^t U_s\,ds\right], \\ U_t = U_0 - \int_0^t \nabla_x P(s,X_s)\,ds + \sigma W_t \\ \textit{with the constraints}: \quad \textit{for all } t\in[0,T],\, \mathbb{P}(X_t\in dx) = dx, \\ \textit{and}\, \partial_x \mathbb{E}[U_t \mid X_t = x] = 0. \end{cases}$$

This solution notably satisfies :

$$Law(X_t, U_t) = f(t,x,u)\,dx\,du.$$

- Particle approximation are valid for the simplified Lagrangian stochastic model.

- Bossy and Violeau 2018 : Quantitative results for particle approximation of simplified Lagrangian stochastic models : For f smooth, $\{K_\epsilon\}$ a sequence of mollifiers and $\epsilon = N^{-\frac{1}{p(d+2)}}$, $1 < p < 1 + \frac{1}{1+3d}$:

$$\mathbb{E}\left[\left|\mathbb{E}[f(U_t)\,|\,X_t] - \frac{\sum_{j=1}^{N} f(U_s^{j,\epsilon,N}) K_\epsilon(X_s^{i,\epsilon,N} - X_s^{j,\epsilon,N})}{\sum_{j=1}^{N} K_\epsilon(X_s^{i,\epsilon,N} - X_s^{j,\epsilon,N})}\right|\right] = \mathcal{O}(N^{-\frac{1}{p(d+2)}}).$$

- Particle approximation are also valid Lagrangian models with specular boundary conditions.

- Maftei 2017 : Reflected Euler scheme for Langevin dynamics with specular boundary condition $\Rightarrow$ Explicit rate of convergence but requires smoothness on the drift coefficient.

- For incompressible : empirical approximation scheme based "Particle in cell methods" show good results (Bernardin et al. 2010), but theoretical validation not done.