On large-time properties of compact Schrödinger semigroups

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Source

Based on joint work with R.L. Schilling (Dresden).

(1) KK, R.L. Schilling: Progressive intrinsic ultracontractivity and heat kernel estimates for non-local Schrödinger operators, *Journal of Functional Analysis* 279 (6), 2020, 108606.

(2) KK, R.L. Schilling: Quasi-ergodicity and the heat content of compact, strong Feller semigroups on L^2, in preparation, 2022

Schrödinger operators with Lévy kinetic terms

- $V : \mathbb{R}^d \to \mathbb{R}$, $V(x) \to \infty$ as $|x| \to \infty \longleftarrow$ confining potential
- $H = -L + V$, $D(H) \subset L^2(\mathbb{R}^d) \longleftarrow$ (self-adjoint) Schrödinger operator

  $L \longleftarrow L^2$-generator of a symmetric strong Feller Lévy process in $\mathbb{R}^d$

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  Brownian case: $L = \Delta$ (local operator)

  jump Lévy case: L is integro-differential operator (non-local operator)

  e.g.

  fractional Laplacian:
  $L = -(-\Delta)^{\alpha/2}$, $\alpha \in (0, 2)$

  relativistic operator:
  $L = -(-\Delta + m^{2/\alpha})^{\alpha/2} + m$, $\alpha \in (0, 2)$, $m \geq 0$

  general case:
  $\widehat{Lf}(\xi) = -\psi(\xi)\widehat{f}(\xi)$, $f \in D(L) := \left\{ f \in L^2(\mathbb{R}^d) : \psi\widehat{f} \in L^2(\mathbb{R}^d) \right\}$

  $$\psi(\xi) = \xi \cdot A\xi + \int_{\mathbb{R}^d \setminus \{0\}} (1 - \cos(\xi \cdot z))\nu(z)dz$$

  A is symmetric non-negative definite $d \times d$ matrix
  $\int (1 \wedge |z|^2)\nu(z)dz < \infty$, $\nu(z)dz$ is infinite and $\nu(z) = \nu(-z)$

A short tour through the assumptions...

(A1) Lévy density $\nu(x)$: there exists a decreasing profile function $f : (0, \infty) \to (0, \infty)$ such that $\nu(x) \asymp f(|x|)$, $x \neq 0$, and

$$\sup_{x \in \mathbb{R}^d} \int \frac{f_1(|x-y|) f_1(|y|)}{f_1(|x|)} dy < \infty, \quad \text{where} \quad f_1 := f \wedge 1.$$

(A2) Heat kernel $e^{tL}(x, y) = p_t(y - x)$ of the free operator L: $(t, x) \mapsto p_t(x)$ is continuous on $(0, \infty) \times \mathbb{R}^d$; $\sup_{x \in \mathbb{R}^d} p_{t_0}(x) = p_{t_0}(0) < \infty$, for some $t_0 > 0$; $p_t(x)$ satisfies a certain time-space upper bound.

(A3) Confining potential: $V \in L^\infty_{\text{loc}}(\mathbb{R}^d)$, $\lim_{|x| \to \infty} V(x) = \infty$ and there is an increasing profile function $g : [0, \infty) \to (0, \infty)$ such that $V(x) \asymp g(|x|)$, $|x| \geq 1$; there is $c \geq 1$ such that $g(r + 1) \leq cg(r)$, $r \geq 1$.

Finally: $g(r) = h(|\log f(r)|)$ for h regular enough (needed for aIUC/non-aIUC).

Schrödinger operators with Lévy kinetic terms (cont.)

- interpretation of $H = -L + V$ (V is confining): *physical oscillator*
  well-known example: $H = -\Delta + |x|^2 \longleftarrow$ Harmonic Oscillator

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- $\sigma(H)$ is discrete, $\lambda_0 = \inf \sigma(H)$

  $\exists! \;\; 0 < \varphi_0 \in L^2(\mathbb{R}^d) \cap C_b(\mathbb{R}^d) \quad H\varphi_0 = \lambda_0 \varphi_0 \longleftarrow$ ground state

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- $\{e^{-tH} : t \geq 0\} \longleftarrow$ Schrödinger semigroup

  $e^{-tH} f(x) = \int_{\mathbb{R}^d} e^{-tH}(x,y) f(y) dy$, $f \in L^2(\mathbb{R}^d)$; $\quad e^{-tH}\varphi_0 = e^{-\lambda_0 t}\varphi_0$

  Problem: what are properties of e^{-tH} as $t \to \infty$?

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- by spectral theorem: $\left\| e^{-t(H-\lambda_0)} f - \left(\int \varphi_0(y) f(y) dy\right) \varphi_0 \right\|_2 \leq e^{-\gamma t} \|f\|_2$

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- $(X_t)_{t \geq 0} \longleftarrow$ process generated by L

$$e^{-tH} f(x) = \mathbb{E}^x \left[\exp\left(- \int_0^t V(X_s) ds \right) f(X_t) \right], \quad f \in L^2(\mathbb{R}^d)$$

(Feynman–Kac formula / semigroup $\longleftarrow$ very useful tool!)
. . . [Demuth-van Casteren]

Intrinsic ultracontractivity for $H = -\Delta + V$

- intrinsic ultracontractivity (IUC): $\forall t > 0 \; \exists c_t \;\; e^{-tH}(x, y) \leq c_t \varphi_0(x)\varphi_0(y)$ [Davies-Simon 1984, Bañuelos 1991]

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$\|\varphi_0\|_2 = 1, \ \mu(dx) = \varphi_0^2(x)dx, \ L^2(\mathbb{R}^d, \mu)$

$\widetilde{H}f = \frac{1}{\varphi_0}(H - \lambda_0)\varphi_0 f, \ f \in D(\widetilde{H}) := \{f \in L^2(\mathbb{R}^d, \mu) : \varphi_0 f \in D(H)\}$

$\left\{e^{-t\widetilde{H}} : t \ge 0\right\} \longleftarrow$ intrinsic (φ_0-transformed) Schrödinger semigroup

$e^{-t\widetilde{H}}f(x) = \int_{\mathbb{R}^d} \frac{e^{-t(H-\lambda_0)}(x,y)}{\varphi_0(x)\varphi_0(y)} f(y)\mu(dy), \ f \in L^2(\mathbb{R}^d, \mu)$

$\{e^{-tH} : t \ge 0\}$ is (IUC) $\Leftrightarrow \left\{e^{-t\widetilde{H}} : t \ge 0\right\}$ is ultracontractive

i.e. $\forall t > 0 \ \ e^{-t\widetilde{H}} : L^2(\mathbb{R}^d, \mu) \to L^\infty(\mathbb{R}^d, \mu)$ bdd

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  $\|\varphi_0\|_2 = 1$, $\mu(dx) = \varphi_0^2(x)dx$, $L^2(\mathbb{R}^d,\mu)$

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- characterisation [Alziary-Takáč, 2009]: $V(x) = V(|x|)$, $V(r) \nearrow \infty$,

$$\{e^{-tH} : t \ge 0\} \text{ is (IUC)} \quad \Longleftrightarrow \quad \int_{r_0}^{\infty} \frac{1}{\sqrt{V(r)}}dr < \infty;$$

  e.g. if $V(x) = |x|^\beta$, then (IUC) holds iff $\beta > 2$.

IUC for non-local Schrödinger operators

- widely studied:

  for $(-\Delta + m^{2/\alpha})^{\alpha/2} - m + V$ and $(-\Delta)^{\alpha/2} + V$:
  Kulczycki-Siudeja 2006; Kwaśnicki 2009; K-Kulczycki 2010; K-Lőrinczi 2012

  for more general non-local kinetic terms:
  K-Lőrinczi 2015; Chen-Wang 2015; 2016

- asymptotic IUC (aIUC in short)[K-Lőrinczi 2012]:
  $\exists t_0, c > 0 \; e^{-t_0 H}(x,y) \leq c\varphi_0(x)\varphi_0(y)$

  $\Leftrightarrow \exists t_0, c > 0 \; \forall t \geq t_0 \; e^{-t_0 H}(x,y) \overset{c}{\asymp} e^{-\lambda_0}\varphi_0(x)\varphi_0(y)$

  $\Leftrightarrow e^{-t\widetilde{H}}(x,y) = \frac{e^{-t(H-\lambda_0)}(x,y)}{\varphi_0(x)\varphi_0(y)} \to 1$ uniformly in $\mathbb{R}^{2d}$ as $t \to \infty$.

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- characterization of aIUC [K-Lőrinczi, 2015]:

  $$\{e^{-tH} : t \geq 0\} \;\text{ is aIUC } \iff \exists c, r > 0 \; \forall |x| \geq r \;\; V(x) \geq c|\log \nu(x)|$$

- estimates of ground state [K-Lőrinczi, 2015]:

  $$\varphi_0(x) \asymp \frac{1 \wedge \nu(x)}{1 \vee V(x)}, \quad x \in \mathbb{R}^d$$

Large-time estimates of heat kernels in non-local settings

Theorem (K-Schilling, 2020)

Under our assumptions on L and V there are $t_0 > 0$ and $R > 1$ such that for $t \geq t_0$ we have the following estimates.

- *If $|x| \wedge |y| \leq R$, then*

$$e^{-tH}(x,y) \asymp e^{-\lambda_0 t} \varphi_0(x) \varphi_0(y).$$

- *If $|x|, |y| > R$, then*

$$e^{-tH}(x,y) \asymp e^{-\lambda_0 t} \varphi_0(x) \varphi_0(y) \ \vee \ \frac{F\left(Ct, x, y\right)}{V(x)V(y)},$$

  where

$$F(\tau, x, y) := \int_{R-1<|z|<|x| \vee |y|} \big(\nu(x-z) \wedge 1\big) \big(\nu(z-y) \wedge 1\big) e^{-\tau V(z)} \, dz$$

(with different C in the lower and the upper bound).

Progressive intrinsic ultracontractivity in non-local settings

Theorem (K-Schilling, 2020)

Under our assumptions on L and V there exist $t_0 > 0$ and an increasing function $\rho : [t_0, \infty) \to (0, \infty]$ such that $\lim_{t\to\infty} \rho(t) = \infty$ and

$$e^{-tH}(x, y) \asymp e^{-\lambda_0 t}\varphi_0(x)\varphi_0(y), \quad |x| \wedge |y| \leq \rho(t),\ t \geq t_0.$$

(comparability constants are uniform in t!)
We call this property progressive intrinsic ultracontractivity (pIUC).

Corollary

$$\sup_{|x|,|y|\leq\rho(t)} \left| \frac{e^{-t(H-\lambda_0)}(x, y)}{\varphi_0(x)\varphi_0(y)} - 1 \right| \leq ce^{-\gamma t}, \quad t \geq t_0.$$

Example 1: $H = (-\Delta)^{\alpha/2} + \log(1 \vee |x|)^\beta$

We have $\nu(x) = c|x|^{-d-\alpha}$, $\alpha \in (0,2)$ and $V(x) = \log(1 \vee |x|)^\beta$, $\beta > 0$.

- $\beta \geq 1$ (aIUC-regime): for $t \geq t_0$ and all $x, y \in \mathbb{R}^d$

$$e^{-tH}(x,y) \asymp \frac{e^{-\lambda_0 t}}{(1+|x|)^{d+\alpha}(1 \vee \log|x|)^\beta (1+|y|)^{d+\alpha}(1 \vee \log|y|)^\beta}.$$

- $\beta \in (0,1)$ (non-aIUC-regime):

for $t \geq t_0$ and $|x| \wedge |y| \leq \rho(t) := \exp\left[C_\beta \left(\frac{t}{d+\alpha}\right)^{\frac{1}{1-\beta}}\right]$,

$$e^{-tH}(x,y) \asymp \frac{e^{-\lambda_0 t}}{(1+|x|)^{d+\alpha}(1 \vee \log|x|)^\beta (1+|y|)^{d+\alpha}(1 \vee \log|y|)^\beta}; \qquad \text{(pIUC)}$$

for $t \geq t_0$ and $|x|, |y| > \rho(t)$, one has

$$e^{-tH}(x,y) \asymp \frac{e^{-\widetilde{C} t (\log(|x| \wedge |y|))^\beta}}{(1+|x-y|)^{d+\alpha}(\log|x| \log|y|)^\beta}$$

(with different $\widetilde{C}$ in the lower and the upper bound).

Example 2: $H = (-\Delta + m^{2/\alpha})^{\alpha/2} - m + |x|^\beta$

$\nu(x) \asymp e^{-m^{1/\alpha}|x|}|x|^{-(d+\alpha+1)/2}$, $|x| \geq 1$, $m > 0$, $\alpha \in (0,2)$; $V(x) = |x|^\beta$, $\beta > 0$.

- $\beta \geq 1$ (aIUC-regime): for $t \geq t_0$ and all $x, y \in \mathbb{R}^d$

$$e^{-tH}(x,y) \asymp e^{-\lambda_0 t} \frac{e^{-m^{1/\alpha}(|x|+|y|)}}{(1+|x|)^{\frac{d+\alpha+1}{2}+\beta}(1+|y|)^{\frac{d+\alpha+1}{2}+\beta}}.$$

- $\beta \in (0,1)$ (non-aIUC-regime):

for $t \geq t_0$ and $|x| \wedge |y| \leq \rho(t) \approx C_{\alpha,m,\beta,d} \cdot t^{\frac{1}{1-\beta}}$,

$$e^{-tH}(x,y) \asymp e^{-\lambda_0 t} \frac{e^{-m^{1/\alpha}(|x|+|y|)}}{(1+|x|)^{\frac{d+\alpha+1}{2}+\beta}(1+|y|)^{\frac{d+\alpha+1}{2}+\beta}}; \quad \text{(pIUC)}$$

for $t \geq t_0$ and $|x|, |y| > \rho(t)$ [$d = 1$ only :-(],

$$e^{-tH}(x,y) \asymp \frac{1}{|x|^\beta |y|^\beta} \left(\frac{e^{-\lambda_0 t - m^{1/\alpha}(|x|+|y|)}}{|x|^{1+\frac{\alpha}{2}}|y|^{1+\frac{\alpha}{2}}} \vee \frac{e^{-\widetilde{C}t(|x|\wedge|y|)^\beta - m^{1/\alpha}|x-y|}}{(1+|x-y|)^{1+\frac{\alpha}{2}}} \right)$$

(with different $\widetilde{C}$ in the lower and the upper bound).

Direct applications to $e^{-tH}\mathbb{1}_{\mathbb{R}^d}$ and spectral regularity

Corollary

There is $t_0 > 0$ and $R > 1$ such that for $t \geq t_0$ we have the following estimates.

- *If $|x| \leq R$, then $e^{-tH}\mathbb{1}_{\mathbb{R}^d}(x) \asymp e^{-\lambda_0 t}$.*
- *If $|x| > R$, then*
  $$e^{-tH}\mathbb{1}_{\mathbb{R}^d}(x) \asymp \frac{1}{g(|x|)} \left(e^{-\lambda_0 t} f(|x|) \ \vee \ \int_{R-1<|z|\leq|x|} f_1(|x-z|) e^{-Ctg(|z|)} \, dz \right)$$
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Definition

e^{-tH} is said to be a/have a

(TC) *trace class operator* if $\int_{\mathbb{R}^d} e^{-tH}(x,x)\,dx < \infty$;

(HS) *Hilbert-Schmidt operator* if $\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-tH}(x,y)^2\,dx\,dy < \infty$;

(fHC) *finite heat content* if $\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-tH}(x,y)\,dx\,dy < \infty$.

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(fHC) *finite heat content* if $\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-tH}(x,y)\,dx\,dy < \infty$.

Corollary

For large times the properties (TC), (HS) and (fHC) coincide and they are equivalent to the condition: there is $t_0 > 0$ such that $\int_{|x|>R} e^{-t_0 V(x)}\,dx < \infty$.

Applications to asymptotic properties

Lemma (K-Schilling, 2022+)

There are $t_0, \gamma, C > 0$ *such that for every* $0 \le s, r < t$ *with* $t - s - r > 2t_0$,

$$\left| e^{-t(H-\lambda_0)}(x,y) - \varphi_0(x)\varphi_0(y) \right| \le C\kappa(t,s,r,x,y), \quad x, y \in \mathbb{R}^d,$$

holds true with the function

$$\kappa(t,s,r,x,y) = \begin{cases} e^{-\gamma t} & \text{if } s = r = 0; \\ e^{-\gamma(t-s)} e^{-s(H-\lambda_0)} \mathbb{1}_{\mathbb{R}^d}(x) & \text{if } s > 0, r = 0; \\ e^{-\gamma(t-r)} e^{-r(H-\lambda_0)} \mathbb{1}_{\mathbb{R}^d}(y) & \text{if } s = 0, r > 0; \\ e^{-\gamma(t-s-r)} e^{-s(H-\lambda_0)} \mathbb{1}_{\mathbb{R}^d}(x) e^{-r(H-\lambda_0)} \mathbb{1}_{\mathbb{R}^d}(y) & \text{if } s > 0, r > 0. \end{cases}$$

- $s = r = 0$ was known: Pinsky 1990, Kim-Song 2008, Zhang-Li-Song 2014
- This extends to very general (also non-self-adjoint) compact semigroups!

Application: for any $p, \widetilde{p} \in [1, \infty]$ and $f \in L^p(\mathbb{R}^d)$, with $c(f) := \int \varphi_0(y) f(y) dy$,

$$\left\| e^{-t(H-\lambda_0)} f - c(f)\varphi_0 \right\|_{\widetilde{p}} \le C e^{-\gamma(t-s-r)} \left\| e^{-s(H-\lambda_0)} \mathbb{1}_{\mathbb{R}^d} \right\|_{\widetilde{p}} \left\| e^{-r(H-\lambda_0)} \mathbb{1}_{\mathbb{R}^d} \right\|_{q} \|f\|_p .$$

Applications to quasi-ergodic properties

Definition

$\widetilde{m} \in \mathcal{M}^1(\mathbb{R}^d)$ is said to be a *quasi-stationary measure* of $\{e^{-tH} : t \geq 0\}$ if for every $f \in L^\infty(\mathbb{R}^d)$ and $t > 0$ we have

$$\frac{\widetilde{m}(e^{-tH} f)}{\widetilde{m}\left(e^{-tH} \mathbb{1}_{\mathbb{R}^d}\right)} = \widetilde{m}(f).$$

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- $m(f) := \frac{\int_{\mathbb{R}^d} f(y)\varphi_0(y)dy}{\|\varphi_0\|_1}$ is a quasi-stationary measure of $\{e^{-tH} : t \geq 0\}$

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- $m(f) := \frac{\int_{\mathbb{R}^d} f(y) \varphi_0(y) dy}{\|\varphi_0\|_1}$ is a quasi-stationary measure of $\{e^{-tH} : t \geq 0\}$

Theorem (K-Schilling, 2022+)

If there exists $t_0 > 0$ such that the heat content is finite,

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-t_0 H}(x,y) dy dx = \int_{\mathbb{R}^d} e^{-t_0 H} \mathbb{1}_{\mathbb{R}^d}(x) dx < \infty,$$

then there are $t_1 > t_0$ and $C > 0$ such that for all $t \geq t_1$, $s \in [t_1, t/2]$ and $f \in L^\infty(\mathbb{R}^d)$

$$\left| \frac{e^{-tH} f(x)}{e^{-tH} \mathbb{1}_{\mathbb{R}^d}(x)} - m(f) \right| \leq C\, e^{-\gamma(t-s)} \, \frac{e^{-s(H-\lambda_0)} \mathbb{1}_{\mathbb{R}^d}(x)}{\varphi_0(x)} \, \|f\|_\infty \,, \quad x \in \mathbb{R}^d.$$

Furthermore, m is unique quasi-stationary measure of $\{e^{-tH} : t \geq 0\}$.

Applications to quasi-ergodic properties (cont.)

Theorem (K-Schilling, 2022+)

If there exists $t_0 > 0$ *and an increasing function* $\rho : [t_0, \infty) \to (0, \infty]$ *such that* $\lim_{t\to\infty} \rho(t) = \infty$ *and*

$$e^{-tH}(x,y) \asymp e^{-\lambda_0 t}\varphi_0(x)\varphi_0(y), \quad |x| \wedge |y| \le \rho(t), \; t \ge t_0, \quad \text{(pIUC)}$$

then for every $a, b \in (0,1)$ *such that* $a + 2b = 1$ *there is* $C > 0$ *such that for all* $t \ge (4/(a \wedge b))t_0$ *and* $f \in L^\infty(\mathbb{R}^d)$

$$\sup_{|x| \le \rho(at)} \left| \frac{e^{-tH} f(x)}{e^{-tH}\mathbb{1}_{\mathbb{R}^d}(x)} - m(f) \right| \le C\, \kappa_b(t)\, \|f\|_\infty ,$$

where $\kappa_b(t) := e^{-\gamma bt} + \sup_{|x| \ge \rho(bt)} e^{-t_0 H}\mathbb{1}_{\mathbb{R}^d}(x)$. *Furthermore, m is unique quasi-stationary measure of* $\left\{e^{-tH} : t \ge 0\right\}$.

- previously known results: aIUC implies uniform on $\mathbb{R}^d$ exponential (in time) quasi-ergodicity (Knobloch-Partzsch 2010, Zhang-Li-Song 2014)

Thank you!