Strong maximum principle for local and non-local Schrödinger operators with singular potential

INdAM Meeting "Kolmogorov Operators and their Applications"

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- A function u on E is called quasi-continuous if for any $\varepsilon > 0$ there exists closed F_ε such that $u_{|F_\varepsilon}$ is continuous and $Cap(E \setminus F_\varepsilon) \leqslant \varepsilon$.

Problem (H. Brezis):

- What is the structure of the set $\{u = 0\}$, where u is a non-trivial function on E that satisfies

$$-Au + V \cdot u \geqslant 0, \qquad u \geqslant 0. \tag{1}$$

The classical strong maximum principle (V is bounded)

E. Hopf (1927) If

$$Au = (a_{i,j}u_{x_i})_{x_j}, \quad \lambda Id \leqslant a \leqslant \Lambda Id, \quad a_{i,j} \in W^{1,\infty}(D), V \in \mathcal{B}_b^+(D)$$

and $u \in C_b^2(D) \cap C(\overline{D})$ is a positive function satisfying

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E. Calabi (1957) considered a weak formulation (prototype of viscosity supersolutions) of (2) and required that u merely be upper semicontinuous (W. Littman observed that if a function satisfies the definition by Calabi then it must also be lower-semicontinuous).

The strong maximum principle (V is bounded or in L^p)

W. Littman (1959) - *A Strong Maximum Principle for Weakly A-Subharmonic Functions* - considered also a weak formulation:

$$-\int_D uA\eta\, dm + \int_D V\eta\, dm \geqslant 0, \quad \eta \in C_c^2(D).$$

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G. Stampacchia (1965): It is enough to assume that $V\in L^p(D)$ for some $p>d/2$.

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with $V(x) := \frac{2d}{|x|^2}$. In this case $\{u = 0\} = \{0\}$, so SMP does not hold.
Observe that $V \in L^1(D)$ for $d \geqslant 3$. For $d = 2$, V is quasi-integrable.

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- Orsina, L., Ponce, A.C.: On the nonexistence of Green's function and failure of the strong maximum principle. *J. Math. Pures Appl.* (2020)

In the paper by Brezis and Bénilan it is assumed that $A = \Delta$ and $V \in L^1_{loc}(\mathbb{R}^d)$. It is shown there that if $u \in L^1_{loc}(\mathbb{R}^d)$ satisfies (1) a.e., then

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$$Au = \sum_{i,j=1}^{d} (a_{ij} u_{x_i})_{x_j}$$

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$$Au = \sum_{i,j=1}^{d} (a_{ij} u_{x_i})_{x_j}$$

on a bounded domain $D \subset \mathbb{R}^d$. It is proved that if a non-trivial (m-a.e.) quasi-continuous $u \in H^1(D)$ (or $u \in L^1(D; m)$ in case a is smooth) satisfies (1) in the sense of measures, then (for q.c. version of u)

$$Cap_2(\{u = 0\}) = 0.$$

Here Cap_2 is the Newtonian capacity.

In the paper by Orsina and Ponce, $A = \Delta_{|D}$ on a bounded domain $D \subset \mathbb{R}^d$ and $V \in L^p(D; m)$ for some $p > 1$. It is proved there that if $u \in L^1(D; m) \cap L^1_V(D; m)$ satisfies (1) in the sense of distributions, and it is non-trivial (m-a.e.), then

$$Cap_{W^{2,p}}(Z) = 0,$$

where

$$Z = \Big\{ x \in D : \limsup_{r \to 0^+} \frac{\int_{B(x,r)} u(y)\, dy}{\ell^d(B(x,r))} = 0 \Big\}.$$

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- A companion problem is the rigorous meaning of the inequality (1). Besides the classical pointwise formulation, one may consider weak formulation for (1):

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for a suitable class $\mathcal{C}$ of test functions, and one may understood (1) in the sense of measures, with the assumption that Au is a Borel measure.

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It appears that the said set N_V admits the following form:

$$N_V := \{x \in E : \nexists\ U_x \text{ - } \textit{finely-open},\ x \in U_x,\ \int_{U_x} G(x,y)V(y)\, m(dy) < \infty\}.$$

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This is an interesting object, which naturally appears in the context of Schrödinger equations with measure data and is well known in the probabilistic potential theory under the name *permanent points* for V.

Theorem (T.K. (2020))

Let $u \in L^1(E;m) \cap L^1_V(E;m)$ *be a positive function satisfying*

$$-\int_E u \cdot A\eta \, dm + \int_E Vu\eta \, dm \geqslant 0, \quad \eta \in \mathcal{C}, \tag{4}$$

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- *there exists an m-version* $\check{u}$ *of* u *which is* finely-continuous *on* $E_V := E \setminus N_V$,
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Theorem

Let $\mathcal{A} := \{u \in L^1(E;m) \cap L^1_V(E;m) : u \geqslant 0,\, u \text{ satisfies } (4),\, u \text{ is finely-continuous and } u \neq 0\}$. Then

$$N_V = \bigcup_{u \in \mathcal{A}} \{u = 0\}.$$

Generalization of the result by Ancona

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Corollary (T.K. (2020))

If u is a non-trivial solution to (4), then

$$\mathcal{H}^G(\{u = 0\}) < \infty.$$

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Remark (Strong maximum principle)

There is a vast literature on estimates of the following form: for any $x \in \mathbb{R}^d$ there exist $\delta_x, c_x > 0$ such that

$$G(x,y) \leqslant c_x \frac{\phi(|x-y|)}{|x-y|^d}, \quad 0 < |x-y| < \delta_x \tag{6}$$

for some continuous increasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, with $\phi(0) = 0$. This combined with the previous corollaries provides an easy method to verify SMP.

Proof technique

Let Ω be the set of functions $\omega : [0, \infty) \to E \cup \{\partial\}$ (where ∂ is a one-point compactification of E in case E is not compact, and an isolated point in case E is compact) such that

1. ω is càdlàg, i.e. it is right continuous and has left limits,
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1. $T_t f(x) = \int_\Omega f(X_t(\omega))\, P_x(d\omega)$ m-a.e.,

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Let Ω be the set of functions $\omega : [0,\infty) \to E \cup \{\partial\}$ (where ∂ is a one-point compactification of E in case E is not compact, and an isolated point in case E is compact) such that

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Let d_S be the Skorochod metric on Ω. With this metric Ω is a separable complete metric space. Let

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As is customary, we denote $\mathbb{E}_x Y := \int_\Omega Y(\omega) \, P_x(d\omega)$ for any $Y \in \mathcal{B}(\Omega)$.

Feynman-Kac formula for supersolutions

Theorem (T.K. (2020))

Assume that $u \in L^1(E;m) \cap L^1_V(E;m)$ *is a positive function that satisfies* (4). *Then there exists an m-version* $\check{u}$ *of* u *which is finely-continuous on* E_V.

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$$\check{u}_k(x) = \mathbb{E}_x\Big[e^{-\int_0^{t\wedge\tau_D} V(X_r)\,dr}\,\check{u}_k(X_{t\wedge\tau_D}) + \int_0^{t\wedge\tau_D} e^{-\int_0^r V(X_s)\,ds}\,dA^{\beta_k}_r\Big],$$

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Theorem (T.K. (2020))

We have

$$x \in N_V \qquad \textit{if and only if} \qquad P_x(\exists_{t>0} \int_0^t V(X_r)\,dr < \infty) = 1.$$

Consider operator

$$-A = \phi(-\Delta), \tag{7}$$

where ϕ is a **Bernstein function** - a smooth function $\phi : (0, \infty) \to [0, \infty)$ such that $(-1)^n D^n \phi \leqslant 0$, $n \geqslant 1$ - i.e.

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Proposition

If A is of the form (7), then the above theorems hold with $\mathcal{C} = C_c^\infty(\mathbb{R}^d)$.

Finely-continuous versions

In general,

$$\check{u}(x) = \limsup_{t \to 0^+} \int_E p(t, x, y) u(y)\, m(dy), \quad x \in E.$$

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$$\check{u}(x) = I^{(\alpha)}(u)(x) := c_{d,\alpha} \limsup_{r\to 0^+} \int_{\mathbb{R}^d \setminus B(x,r)} \frac{r^\alpha u(y)}{|x-y|^d(|x-y|^2 - r^2)^\alpha}\,dy.$$

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The above set functions are called Riesz capacities.

Generalization of Ponce and Orsina's result

Theorem

Let V be a positive Borel function such that $V \in L^p(E;m)$ for some $p \geqslant 1$. Assume that $u \in L^1(E;m) \cap L^1_V(E;m)$ is a positive function such that

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If $A = \Delta$, then $C_p \sim Cap_{W^{2,p}}$ for $p > 1$.

The result by Bertsch et al.

Let D be an open interval and $c : D \to [0, +\infty]$. Set

$$\mathcal{N} := \{x \in D : \nexists \delta > 0 \text{ such that } c \in L^1(x - \delta, x + \delta)\}.$$

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Theorem (Bertsch et al. (2015))

The following statements are equivalent:

- *SMP holds for* $-\frac{d^2}{dx^2} + c$ *(i.e. if* $u \in C^+(D), c \cdot u \in L^1_{loc}(D)$*,* $-u'' + c \cdot u \geqslant 0$ *in* D*, and* $u(x) = 0$ *for some* $x \in D$*, then* $u \equiv 0$*);*

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Observe that $\mathcal{N} = N_V$ with $V := c$. This is so because in one dimension fine topology and Euclidean topology agree.

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In case $\mathcal{N} \neq \emptyset$, c is not quasi-integrable. In this case we also have that any positive solution to $-u'' + c \cdot u \geqslant 0$ in D is trivial.

The general fact

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Theorem (Orsina and Ponce, 2008)

There is no -non-trivial positive solutions to

$$-\Delta u + V \cdot u \geqslant 0 \quad \text{in } B(0,1),$$

where $V(x) = \frac{1}{|x_1|^\alpha}$, with $\alpha \in [1, 2)$.

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