Liouville-type theorems for Kolmogorov-Fokker-Planck and Ornstein-Uhlenbeck operators

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The results presented are contained in a series of papers in collaboration with A.Bonfiglioli, E.Lanconelli, Y.Pinchover, S.Polidoro, E.Priola

Kolmogorov-Fokker-Planck operators

$$\mathcal{L} = \mathrm{div}(A\nabla) + \langle Bx, \nabla \rangle - \partial_t, \text{ in } \mathbb{R}^{N+1} = \mathbb{R}^N_x \times \mathbb{R}_t$$

$A = (a_{ij})_{i,j=1,\dots,N}$ and $B = (b_{ij})_{i,j=1,\dots,N}$ are $N \times N$ matrices with real constant coefficients

A is symmetric and non-negative definite

A and B satisfy additional suitable conditions in order that the operator $\mathcal{L}$ is hypoelliptic

The following conditions on the operator $\mathcal{L}$ are equivalent:

(i) The operator $\mathcal{L}$ is hypoelliptic.

(ii) There exist bases of $\mathbb{R}^N$ such that the matrix A takes the following block form:

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix},$$

for some $p_0 \times p_0$ symmetric and strictly positive definite matrix A_0, $p_0 \leq N$. Moreover, if $p_0 < N$, the matrix B can be written as follows

$$B = \begin{bmatrix} * & * & \dots & * & * \\ B_1 & * & * & * & * \\ 0 & B_2 & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & B_n & * \end{bmatrix}, \tag{1}$$

where B_j is a $p_j \times p_{j-1}$ matrix with maximum rank p_j; $j = 1, 2, ..., n$, $p_0 \geq p_1 \geq ... \geq p_n \geq 1$ and $p_0 + p_1 + ... + p_n = N$. Every block $*$ is a real constant matrix that does not need to satisfy any particular condition.

(iii) The first order differential operators

$$X_i = \sum_{k=1}^{N} a_{ik} \partial_{x_k}, \quad i = 1, \ldots, N, \quad \text{and} \quad Y = \langle Bx, \nabla \rangle - \partial_t,$$

satisfy the *Hörmander condition*

$$\mathrm{rank\ Lie}\{X_1, \ldots, X_N, Y\}(z) = N + 1 \quad \forall z \in \mathbb{R}^{N+1},$$

(iv) Letting

$$E(s) := e^{-sB}, \quad s \in \mathbb{R}, \tag{2}$$

the matrix

$$C(t) = \int_0^t E(s) A E^T(s)\, ds$$

satisfies the *Kalman condition*, that is, C is strictly positive definite for every $t > 0$.

Example: Kolmogorov operator

Suppose that the matrices A and B are of the type:

$$A = \begin{pmatrix} \mathbb{I}_n & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 0 \\ \mathbb{I}_n & 0 \end{pmatrix}.$$

$$\mathcal{L} = \mathrm{div}(A\nabla) + \langle Bx, \nabla \rangle - \partial_t = \sum_{i=1}^{n} \partial_{x_i}^2 + \sum_{i=1}^{n} x_i \partial_{x_{n+i}} - \partial_t \quad \text{in } \mathbb{R}^{2n+1}.$$

In this case:

$$C(t) = \int_0^t e^{-sB}\, A\, e^{sB^T}\, ds = \begin{pmatrix} t\mathbb{I}_n & -\frac{t^2}{2}\mathbb{I}_n \\ -\frac{t^2}{2}\mathbb{I}_n & \frac{t^3}{3}\mathbb{I}_n \end{pmatrix} > 0, \quad \forall t > 0.$$

$$\mathcal{L} = \mathrm{div}(A\nabla) + \langle Bx, \nabla\rangle - \partial_t, \text{ in } \mathbb{R}^{N+1} = \mathbb{R}^N_x \times \mathbb{R}_t$$

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} * & * & * & \dots & * \\ B_1 & * & * & \dots & * \\ 0 & B_2 & \dots & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & B_n & * \end{pmatrix}$$

A_0 $\quad p_0 \times p_0$ matrix

A symmetric and positive semidefinite

B_j $\quad p_j \times p_{j-1}$ matrix with rank p_j

$$p_0 \geq p_1 \geq \ldots \geq p_n \geq 1$$

$$p_0 + p_1 + \ldots + p_n = N$$

Lanconelli and Polidoro proved that $\mathcal{L}$ is left translation invariant with respect to the Lie group $\mathbb{K} = (\mathbb{R}^{N+1}, \circ)$ with composition law

$$(x,t) \circ (x', t') = (x' + E(t')x, t + t'), \quad \text{where} \quad E(t) = e^{-tB}$$

Other class of operators satisfying L^p-Liouville property

Heat-type operators on stratified Lie groups:

$$\mathcal{L} = \mathcal{L}_0 - \partial_t := \sum_{j=1}^{m} X_j^2 - \partial_t \quad \text{in } \mathbb{R}^{N+1}$$

$X_1, \ldots, X_m$ are smooth first order linear PDOs generating the Lie algebra of a stratified Lie group in $\mathbb{R}^N$

prototype: Heat operator on the Heisenberg group

$$\mathcal{L} = \Delta_{\mathbb{H}_1} - \partial_t := (\partial_{x_1} + 2x_2\partial_{x_3})^2 + (\partial_{x_2} - 2x_1\partial_{x_3})^2 - \partial_t \quad \text{in } \mathbb{R}^4$$

sub-Kolmogorov-type operators:

$$\mathcal{L} := \sum_{j=1}^{m} X_j^2 + \langle Bx, \nabla \rangle - \partial_t \quad \text{in } \mathbb{R}^{N+1}$$

prototype:

$$\mathcal{L} = \Delta_{\mathbb{H}_1} + x_1\partial_{x_4} - \partial_t := (\partial_{x_1} + 2x_2\partial_{x_3})^2 + (\partial_{x_2} - 2x_1\partial_{x_3})^2 + x_1\partial_{x_4} - \partial_t \quad \text{in } \mathbb{R}^5$$

- *Fokker-Planck equations of Mumford type:*

$$\mathcal{L} = \partial_{x_1}^2 + \sin x_1 \partial_{x_2} + \cos x_1 \partial_{x_3} - \partial_t \quad \text{in } \mathbb{R}^4$$

Some basic references:

- *Kinetic Fokker-Planck equations* Helffer-Nier (2005)
- *Kolmogorov operators of stochastic equations* Da Prato (2004)
- *PDEs model in finance* Pascucci (2005)
- *Computer and human vision* Mumford (1994)
- *Curvature Brownian motion* Chirikjian, Maslen, Wang and Zhou (2006)
- *Phase-noise Fokker-Planck equations* August and Zucker (2003)

$$\mathcal{L}u = 0 \text{ in } \mathbb{R}^{N+1},\ u \in L^p(\mathbb{R}^{N+1}) \implies u \equiv 0$$

We also show L^p-Liouville type theorems for solutions to

$$\mathcal{L}u \geq 0 \text{ in } \mathbb{R}^{N+1}.$$

$$\mathcal{L}u \geq 0 \text{ in } \mathbb{R}^{N+1},\ u \in L^p(\mathbb{R}^{N+1}),\ u \geq 0 \implies u = 0 \text{ a.e.}$$

These results are useful to give necessary conditions for semilinear equations like

$$\mathcal{L}u = f(u) \text{ in } \mathbb{R}^{N+1}$$

have *non-trivial solutions.*

To show our results we use:

- the mean value characterization of caloric functions
- a Poisson-Jensen formula for sub-caloric functions
- some results and devices from Parabolic Potential Theory

- *Fundamental solution for* $\mathcal{H} = \Delta_{\mathbb{R}^N} - \partial_t$

$$\Gamma : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}, \quad \Gamma(x,t) = \begin{cases} 0 & \text{if } t \leq 0 \\ (4\pi t)^{\frac{-N}{2}} e^{-\frac{|x|^2}{4t}} & \text{if } t > 0 \end{cases}$$

- *Heat ball* of center $z = (x,t)$ and radius r

$$\Omega_r(z) = \left\{ \zeta \in \mathbb{R}^{N+1} \ : \ \Gamma(z - \zeta) > \frac{1}{r} \right\}$$

- *Watson kernel*

$$K_r(z) = K_r(x,t) = \frac{1}{r} \frac{|\nabla_x \Gamma(x,t)|^2}{\Gamma^2(x,t)} = \frac{1}{r} \left(\frac{|x|}{2t} \right)^2$$

- *Caloric functions*
  A caloric function in an open set $O \subseteq \mathbb{R}^{N+1}$ is a function $u \in C^\infty(O, \mathbb{R})$ such that

$$\mathcal{H}u = 0 \ \text{ in } O.$$

- *Pini-Fulks-Watson Theorem*
  If u is caloric in O then

$$u(z) = M_r(u)(z) := \int_{\Omega_r(z)} u(\zeta) K_r(z-\zeta)\, d\zeta \qquad \forall\ \overline{\Omega_r(z)} \subseteq O. \quad (*)$$

  Viceversa: if $u \in C(O, \mathbb{R})$ satisfies $(*)$, then

$$u \in C^\infty(O, \mathbb{R}) \quad \text{and} \quad \mathcal{H}(u) = 0 \text{ in } O.$$

- *Sub-caloric functions*
  A function $u : O \longrightarrow [-\infty, \infty[$ is *sub-caloric* if:
  (i) u is upper semicontinuous;
  (ii) $u > -\infty$ in a dense subset in O;
  (iii) $u(z) \le M_r(u)(z) \qquad \forall\ \overline{\Omega_r(z)} \subseteq O$.

Proposition Let $u : O \longrightarrow [-\infty, \infty[$ be u.s.c.
Then u is sub-caloric in O if and only if

$$\boxed{u \in L^1_{\mathrm{loc}}(O), \quad \mathcal{H}u \ge 0 \text{ in } \mathcal{D}'(O),} \quad u(z) = \lim_{r \searrow 0} M_r(u)(z). \qquad (**)$$

Moreover, if $u \in L^1_{\mathrm{loc}}(O)$ is a weak solution to $\mathcal{H}u \ge 0$,
there exists a sub-caloric function $\tilde{u}$ in O s.t.

$$u(z) = \tilde{u}(z) \qquad \text{a.e. in } O.$$

remark By Riesz-Schwartz Theorem, if u is sub-caloric there exists a

nonnegative Radon measure μ in $\mathbb{R}^{N+1}$ such that $\mathcal{H}u = \mu$.

- *Caloric Poisson-Jensen formula*
  If u is *sub-caloric* in $\mathbb{R}^{N+1}$ and $\mu = \mathcal{H}u$, then

$$u(z) = M_r(u)(z) - N_r(\mu)(z) \qquad \forall\ z \in \mathbb{R}^{N+1},$$

where M_r is the Watson average operator and

$$N_r(\mu)(z) := \frac{1}{r}\int_0^r \left(\int_{\Omega_\rho(z)} \left(\Gamma(z-\zeta) - \frac{1}{\rho} \right)\ d\mu(\zeta) \right)\ d\rho.$$

proposition

$$N_r(\mu)(z) = 0 \quad \forall\ z \in T,\ \overline{T} = \mathbb{R}^{N+1} \implies \mu \equiv 0$$

Theorem 1 Let $u \in C^\infty(\mathbb{R}^{N+1})$ be a caloric function

$$\mathcal{H}u = 0 \quad \text{in } \mathbb{R}^{N+1}.$$

Suppose $u \in L^p(\mathbb{R}^{N+1})$ for a suitable $p \in [1, \infty[$.

$$\text{Then} \qquad u \equiv 0.$$

— —

remark Let $u \in C^\infty(\mathbb{R}^N)$ be a harmonic function $\Delta u = 0$ in $\mathbb{R}^N$. If $u \in L^p(\mathbb{R}^N)$ and $1 \le p < \infty$, for every x in $\mathbb{R}^N$:

$$|u(x)| = \left| \fint_{B_r(x)} u(y)\; dy \right| \le \left(\frac{1}{|B_r(x)|} \right)^{\frac{1}{p}} \|u\|_{L^p(\mathbb{R}^N)} \longrightarrow 0 \quad \text{as } r \longrightarrow \infty$$

Our new approach is based on the caloric Poisson-Jensen formula

$$u = M_r(u) - N_r(\mu). \tag{PJ}$$

sketch of the proof of Theorem 1

(I) Let $u \in L^1(\mathbb{R}^{N+1})$ be such that $\mathcal{H}u \geq 0$ in $\mathbb{R}^{N+1}$. Then,

$$\mathcal{H}u \equiv 0 \text{ in } \mathbb{R}^{N+1}.$$

An easy exchange of integrals shows that

$$\boxed{\int_{\mathbb{R}^{N+1}} M_r(u)(z)\, dz = \int_{\mathbb{R}^{N+1}} u(z)\, dz} \qquad \forall r > 0.$$

Then, from (PJ) we get

$$\int_{\mathbb{R}^{N+1}} N_r(\mu)(z)\, dz = 0 \qquad \forall r > 0.$$

Since $\mu = \mathcal{H}u \geq 0$ everywhere, this gives

$$N_r(\mu)(z) = 0 \text{ a.e. in } \mathbb{R}^{N+1} \implies \mu \equiv 0 \implies \mathcal{H}u \equiv 0 \text{ in } \mathbb{R}^{N+1}$$

(II) If $u \in C^2(\mathbb{R}^{N+1}, \mathbb{R})$ and $F \in C^2(\mathbb{R}, \mathbb{R})$, then

$$\mathcal{H}(F(u)) = F'(u)\mathcal{H}(u) + F''(u)|\nabla_x u|^2$$

(III) Let $\mathcal{H}u = 0$ in $\mathbb{R}^{N+1}$ and

$$u \in L^p(\mathbb{R}^{N+1}),\ 1 \leq p < \infty.$$

Define

$$v := F(u)$$

where

$$F : \mathbb{R} \longrightarrow \mathbb{R}, \quad F(t) = (\sqrt{1+t^2} - 1)^p = \left(\frac{t^2}{\sqrt{1+t^2}+1}\right)^p.$$

Since

$$0 \leq F(t) \leq |t|^p \quad \text{and} \quad F''(t) > 0 \quad \forall t \neq 0,$$

we have

$$0 \leq v \leq |u|^p \quad \Longrightarrow \quad v \in L^1(\mathbb{R}^{N+1})$$

$$\mathcal{H}(v) = F''(u)|\nabla_x u|^2 \geq 0 \quad \text{in } \mathbb{R}^{N+1}.$$

Then, $\mathcal{H}(v) \equiv 0$, i.e.,

$$F''(u)|\nabla_x u|^2 = 0 \quad \text{in } \mathbb{R}^{N+1}$$

$$\Downarrow$$

$$|\nabla_x u|^2 = 0 \quad \text{in } U_0 = \{u \neq 0\}$$

$$\Downarrow$$

$$\Delta u = 0 \quad \text{in } U_0 \qquad \Longrightarrow (\mathcal{H}u = 0) \ \partial_t u = 0 \quad \text{in } U_0$$

$$\Downarrow$$

$$|\nabla_z u| = 0 \quad \text{in } U_0$$

$$\Downarrow$$

$$u \equiv 0 \quad \text{in } \mathbb{R}^{N+1}$$

Theorem 2 Let $u \in L^p(\mathbb{R}^{N+1})$, $1 \le p < \infty$, be a *nonnegative* solution of

$$\mathcal{H}u \ge 0 \quad \text{in } \mathbb{R}^{N+1}.$$

Then

$$u = 0 \quad \text{a.e.} \quad \text{in } \mathbb{R}^{N+1}.$$

$\mathbb{G} = (\mathbb{R}^{N+1}, \circ)$ H= *right invariant Haar measure* on $\mathbb{G}$

$$L^p(\mathbb{R}^{N+1}, H), \quad L^p\text{-space with respect to } H$$

Theorem 1 Let $u \in C^\infty(\mathbb{R}^{N+1}, \mathbb{R})$ be a solution to

$$\mathcal{L}u = 0 \ \text{ in } \mathbb{R}^{N+1}.$$

Then $u \equiv 0$ if one of the following conditions is satisfied

(i) $u \in L^p(\mathbb{R}^{N+1}, H)$ for a suitable $p \in [1, \infty[$;

(ii) $u \geq 0$ and $u^p \in L^1(\mathbb{R}^{N+1}, H)$ for a suitable $p \in]0, 1[$.

Theorem 2 Let $u \in C^2(\mathbb{R}^{N+1}, \mathbb{R})$ be a solution to

$$\mathcal{L}u \geq 0 \ \text{ in } \mathbb{R}^{N+1}.$$

If $u \in L^p(\mathbb{R}^{N+1}, H)$ for a suitable $p \in [1, \infty[$, then $u \leq 0$.

$\mathbb{G} = (\mathbb{R}^{N+1}, \circ)$ *H= right invariant Haar measure on* $\mathbb{G}$
given $x \in \mathbb{R}^N$ we set

$$\sigma_x : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}, \quad \sigma_x(y) := y \circ x$$

$$E \mapsto H(E) := \int_E \frac{1}{\det\left(\mathcal{J}\sigma_x(0)\right)}\, dx$$

Kolmogorov-Fokker-Planck operators

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A and B constant $N \times N$ real matrices

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} * & * & * & \dots & * \\ B_1 & * & * & \dots & * \\ 0 & B_2 & \dots & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & B_n & * \end{pmatrix}$$

A_0 $p_0 \times p_0$ matrix

A symmetric and positive semidefinite

B_j $p_j \times p_{j-1}$ matrix with rank p_j

$$p_0 = p \geq p_1 \geq \dots \geq p_n \geq 1$$

$$p_0 + p_1 + \dots + p_n = N$$

$$E(t) := \exp(-tB), \quad t \in \mathbb{R}$$

$$\mathbb{G} = (\mathbb{R}^{N+1}, \circ) \qquad (x,t) \circ (x',t') = (x' + E(t')x, t + t')$$

$$H(E) := \int_E \frac{1}{\det\left(\mathcal{J}_{\sigma_{z'}}(0)\right)}\, dz' \qquad \mathcal{J}_{\sigma_{z'}}(0) = \begin{pmatrix} E(t') & 0 \\ 0 & 1 \end{pmatrix}$$

$$\det(E(t)) = \exp(-t\,\mathrm{trace}(B)) \implies dH(x,t) = e^{t\,\mathrm{trace}(B)}\, dxdt$$

$$L^p(\mathbb{R}^{N+1}, H) = L^p(\mathbb{R}^{N+1}, e^{t\,\mathrm{trace}(B)}\, dxdt)$$

Theorem

Let u be a solution to $\mathcal{L}u = 0$ in $\mathbb{R}^{N+1}$. If

$$\int_{\mathbb{R}^{N+1}} |u(x,t)|^p \, e^{t \, trace(B)} \; dxdt < \infty$$

for some $p \in [1, \infty)$, then $u \equiv 0$.

Theorem

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for some $p \in [1, \infty)$, then $u \leq 0$ in $\mathbb{R}^{N+1}$.

Uniqueness for the Cauchy Problem

$$\begin{cases} \partial_t u = \mathcal{L}_0 u & \text{in } \mathbb{R}^{N+1}_+ := \mathbb{R}^N \times]0, \infty[\\ u|_{t=0} = 0 \end{cases} \tag{PC}$$

if u is a solution of (PC) and for some $p \in [1, \infty)$

$$\int_0^\infty \int_{\mathbb{R}^n} |u(x,t)|^p \, e^{t \, trace(B)} \, dxdt < \infty,$$

then $u \equiv 0$.

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$$p_0 \geq p_1 \geq \ldots \geq p_n \geq 1$$

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$\widetilde{\mathcal{L}}$ is homogeneous of degree two with respect to the group of dilations

$$\delta_r : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}^{N+1}, \quad \begin{aligned} \delta_r(x,t) &= \delta_r(x^{(p_0)}, x^{(p_1)}, \ldots, x^{(p_n)}, t) \\ &:= (rx^{(p_0)}, r^3 x^{(p_1)}, \ldots, r^{2n+1} x^{(p_n)}, r^2 t), \end{aligned}$$

where $x^{(p_i)} \in \mathbb{R}^{p_i}$, $i = 0, \ldots, n$, and $r > 0$.

Let $Q := p_0 + 3p_1 + ... + (2n+1)p_n + 2$. The number Q denotes the *homogeneous dimension* of $\widetilde{\mathcal{L}}$ with respect to the group of dilations $(\delta_r)_{r>0}$. We can state the following theorem.

Theorem

Let $u \in L^1_{loc}$ be a solution to $\widetilde{\mathcal{L}}u \geq 0$ in $\mathbb{R}^{N+1}$, in the sense of distributions. Let $u \in L^p(\mathbb{R}^{N+1})$, $1 \leq p < \infty$. Then, $u \leq 0$ in $\mathbb{R}^{N+1}$. Furthermore, if there exists $p \in [1, 1 + \frac{2}{Q-2}]$ such that

$$\int_{\mathbb{R}^{N+1}} |u(x,t)|^p \, dxdt < \infty,$$

then

$$u \equiv 0 \quad a.e. \ \ in \ \mathbb{R}^{N+1}.$$

Moreover, for every $p > 1 + \frac{2}{Q-2}$, there exists $u \in L^p(\mathbb{R}^{N+1})$, $u \leq 0$, $u \not\equiv 0$, such that

$$\widetilde{\mathcal{L}}u \geq 0 \ \ in \ \mathbb{R}^{N+1}, in\ the\ sense\ of\ distributions.$$

The lack of the *non-negative Liouville Theorem for the Heat operator*

$$\mathcal{L} = \mathcal{H} := \Delta - \partial_t \quad \text{in } \mathbb{R}^{N+1}$$

$$\Delta = \sum_{j=1}^{N} \partial^2_{x_j} \qquad \text{and} \qquad z = (x,t) = (x_1, \ldots, x_N, t)$$

$$\boxed{\mathcal{H}u = 0 \text{ in } \mathbb{R}^{N+1},\ u \geq 0 \ \not\Longrightarrow\ u \equiv const}$$

$u(x,t) = u(x_1, \ldots, x_N, t) = e^{x_1 + \ldots + x_N + Nt}$
satisfies
$\mathcal{H}u = 0, \quad u \geq 0 \quad \text{in } \mathbb{R}^{N+1}$

u is **not** constant

Liouville Theorem at "$t = -\infty$"

Let u be a nonnegative solution to the equation

$$\widetilde{\mathcal{L}} u = 0$$

in the halfspace $S = \mathbb{R}^N \times] - \infty, 0[$.

Then

$$\lim_{t \longrightarrow -\infty} u(x,t) = \inf_S u \qquad \forall x \in \mathbb{R}^N$$

$$u \text{ is } \textbf{constant at } t = -\infty$$

L^∞-Liouville Theorem for $\widetilde{\mathcal{L}}$

Let $\widetilde{\mathcal{L}}u = 0$ in $\mathbb{R}^{N+1}$ and

$$u \in L^\infty(\mathbb{R}^{N+1}).$$

Define

$$m = \inf_{\mathbb{R}^{N+1}} u \quad \text{and} \quad M = \sup_{\mathbb{R}^{N+1}} u.$$

Then,

$$u - m \quad \text{and} \quad M - u$$

are *nonnegative caloric* functions in $\mathbb{R}^{N+1}$.

$$\lim_{t \longrightarrow -\infty} (u(x,t) - m) = \inf_{\mathbb{R}^{N+1}} (u - m) = 0, \quad \forall x \in \mathbb{R}^N;$$

$$\lim_{t \longrightarrow -\infty} (M - u(x,t)) = \inf_{\mathbb{R}^{N+1}} (M - u) = 0, \quad \forall x \in \mathbb{R}^N.$$

As a consequence,

$$M - m = (M - u(x,t)) + (u(x,t) - m) \longrightarrow 0 \text{ as } t \longrightarrow -\infty,$$

that is $M - m = 0$, and *u is constant.*

Asymptotic Liouville Theorem for $\widetilde{\mathcal{L}}$

Let $u : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}$ be a smooth function s. t.

$$\begin{cases} \widetilde{\mathcal{L}} u = 0 & \text{in } \mathbb{R}^{N+1} \\ u \geq 0 & \text{in } \mathbb{R}^{N+1} \end{cases},$$

Assume

$$u(0,t) = O(t^m) \qquad \text{as } t \longrightarrow \infty$$

Then, $u \equiv$ constant.

– –

remark In the particular case that B is the zero matrix, the asymptotic estimate $u(0,t) = O(t^m)$ can be replaced by

$$u(0,t) = O(e^{\varepsilon} t) \text{ as } t \longrightarrow \infty,$$

for every $\varepsilon > 0$.

Polynomial Liouville Theorem for $\widetilde{\mathcal{L}}$

Let $u : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}$ be a smooth function s. t.

$$\begin{cases} \widetilde{\mathcal{L}} u = p & \text{in } \mathbb{R}^{N+1} \\ \\ u \geq q & \text{in } \mathbb{R}^{N+1} \end{cases},$$

where p and q are polynomial functions.
Assume

$$u(0,t) = O(t^m) \qquad \text{as } t \longrightarrow \infty$$

Then, u is a polynomial function.

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where p and q are polynomial functions.
Assume

$$u(x,0) = O(|x|^m) \qquad \text{as } |x| \longrightarrow \infty$$

Then, u is a polynomial function.

Moreover, if p and q are identically zero,

$$u \equiv \text{constant}$$

Polynomial Liouville Theorem for $\widetilde{\mathcal{L}}_0 = \mathrm{div}(A\nabla) + \langle Bx, \nabla \rangle$

Let $P, Q : \mathbb{R}^N \longrightarrow \mathbb{R}$ be polynomial functions
and
let $U : \mathbb{R}^N \longrightarrow \mathbb{R}$ be a smooth function s. t.

$$\widetilde{\mathcal{L}}_0 U = P \quad \text{and } U \geq Q, \quad \text{in } \mathbb{R}^N.$$

Then U is a polynomial function.

– –

We remark that our class of operators $\widetilde{\mathcal{L}}_0$ also contains “parabolic” type operators like, e.g. the following “forward-backward” heat operator

$$\widetilde{\mathcal{L}}_0 := \partial_{x_1}^2 + x_1 \partial_{x_2} \qquad \text{in } \mathbb{R}^2.$$

One-side Liouville Theorem for $\widetilde{\mathcal{L}}_0 = \mathrm{div}(A\nabla) + \langle Bx, \nabla\rangle$

Let $U : \mathbb{R}^N \longrightarrow \mathbb{R}$ be a smooth function s. t.

$$\widetilde{\mathcal{L}}_0 U = 0 \quad \text{and } U \geq 0, \quad \text{in } \mathbb{R}^N.$$

Then $U \equiv$ *constant* .

Proof

Let us define $u(x,t) = U(x)$. Then u is a non-negative solution to the equation $\widetilde{\mathcal{L}} u = 0$ in $\mathbb{R}^{N+1}$. Moreover, $u(0,t) = U(0) = O(1)$ as $t \longrightarrow \infty$. Then, u is constant, and so U is constant.

L^∞-Liouville Theorems

Theorem

Let u be a bounded solution to $\widetilde{\mathcal{L}}u = 0$ in $\mathbb{R}^{N+1}$. Then, u is constant.

Corollary

Let U be a bounded solution to $\widetilde{\mathcal{L}}_0 U = 0$ in $\mathbb{R}^N$. Then, U is constant.

Actually this last result is a particular case of a theorem due to Priola and Zabczyk that holds under more general hypotheses on the matrix B. More precisely, the Priola and Zabczyk theorem for the operator $\mathcal{L}_0$, takes this form.

Any bounded solution v to $\mathcal{L}_0 v = 0$ in $\mathbb{R}^N$ is constant

$$\Longleftrightarrow$$

the real part of every eigenvalue of the matrix B is non-positive.

It is an open problem if in the general case, where all the eigenvalues of the matrix B have non positive real parts, the solutions to the equation $\mathcal{L}_0 v = 0$ bounded *just from one side* have to be constant.

Very recently, with Priola and Lanconelli, we obtained the one-side Liouville property for operators of the type

$$\sum_{i=1}^{N} \partial_{x_i}^2 u \ + \langle Bx, \nabla \rangle,$$

when B is diagonalizable over the complex field with all its eigenvalues on the imaginary axis.

An historical note Actually, Augustin-Louis Cauchy was the first to announce and prove, on the 23rd of December 1844, the earliest statement of the theorem nowadays known as the "the Liouville theorem",

any bounded entire function of a single complex variable has to be constant,

two weeks after Joseph Liouville, on the 9th of December, announced the result in the special case of doubly periodic functions,

a doubly-periodic holomorphic function has to be constant.

Both the results were published in the Comptes Rendus de l'Académie des Sciences, Paris.

Lützen in the chapter *The Discovery of Liouville's Theorem* asserts that "Liouville justly deserves the honor of having his name attached to the theorem for the following three reasons.
1° Liouville was the first to publish the theorem in the doubly periodic case, Cauchy's 1843 theorems being clearly different from, although closely related to, Liouville's theorem.
2° Liouville was the first to discover the fundamental importance of the theorem.
3° Liouville probably had arrived at the general form of Liouville's theorem before Cauchy."
Contrarily, Serrin and Zou, in their historical note *Cauchy and Liouville, a question of priority* argue that "Lützen's presentation is partly marred by championship of Liouville's priority claims".