A small time approximation for the solution to the Zakai Equation

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Introduction

The object of our investigation is the Cauchy problem

$$\begin{cases} du(t,x) = \mathcal{A}_x u(t,x)dt + h(x)u(t,x)dB_t, & t \in [0,T], x \in \mathbb{R}^d; \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases}$$

with

$$\mathcal{A}_x := \sum_{i,j=1}^{d} \alpha_{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{d} \beta_i(x)\partial_{x_i} + \gamma(x).$$

Here

- the functions $\alpha_{ij}, \beta_i, \gamma : \mathbb{R}^d \to \mathbb{R}$ are smooth and bounded for all $i,j = 1, ..., d$;
- the matrix $\{\alpha_{ij}(x)\}_{1 \leq i,j \leq d, x \in \mathbb{R}^d}$ is uniformly elliptic

$$\mu_1 |z|^2 \leq \sum_{i,j=1}^{d} \alpha_{ij}(x) z_i z_j \leq \mu_2 |z|^2, \quad \text{for all } z \in \mathbb{R}^d;$$

- the functions $h, u_0 : \mathbb{R}^d \to \mathbb{R}$ are bounded and continuous;
- $\{B_t\}_{t \in [0,T]}$ is a one dimensional Brownian motion.

The filtering problem

- Signal process (d dimensional)

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t m(X_s)dB_s, \quad t \in [0, T],$$

  with $\{B_t\}_{t\in[0,T]}$ being a standard d-dimensional Brownian motion.

- Observation process (one dimensional)

$$Y_t = y_0 + \int_0^t h(X_s)ds + W_t, \quad t \in [0, T],$$

  with $\{W_t\}_{t\in[0,T]}$ being a standard one-dimensional Brownian motion independent of $\{B_t\}_{t\in[0,T]}$.

According to the general theory of conditional expectations, the best $\mathbb{L}^2(\Omega)$-prediction of $\varphi(X_t)$ given the information $\mathcal{F}_t^Y := \sigma(Y_s, 0 \leq s \leq t)$ is

$$\mathbb{E}[\varphi(X_t)|\mathcal{F}_t^Y] = \int_{\mathbb{R}^d} \varphi(x)\pi_t(dx) =: \pi_t(\varphi).$$

- Problem: find/characterize the filtering distribution $\{\pi_t\}_{t\in[0,T]}$.

The filtering problem

Theorem (Kushner 1967, Fujisaki-Kallianpur-Kunita 1972)

For any $\varphi \in C_b^2(\mathbb{R}^d)$ *the stochastic process* $\{\pi_t(\varphi)\}_{t\in[0,T]}$ *solves*

$$\begin{cases} d\pi_t(\varphi) = \pi_t(\mathcal{L}_x\varphi)dt + [\pi_t(h\varphi) - \pi_t(h)\pi_t(\varphi)]\,(dY_t - \pi_t(h)dt)\,; \\ \pi_0(\varphi) = \mathbb{E}[\varphi(X_0)], \end{cases} \tag{1}$$

where $\mathcal{L}_x$ *is the generator of the signal* $X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t m(X_s)dB_s$, *i.e.*

$$\mathcal{L}_x := \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^d b_i(x)\partial_{x_i} \quad \text{with} \quad a_{ij}(x) := \sum_{k=1}^d m_{ik}(x)m_{jk}(x),$$

while the function h *comes from the observation process*

$$Y_t = y_0 + \int_0^t h(X_s)ds + W_t, \quad t \in [0, T].$$

Equation (1) admits, under suitable assumptions, a unique solution and thus identifies the filtering distribution $\{\pi_t\}_{t\in[0,T]}$. However, the complex structure of equation (1) makes its analysis quite demanding.

The filtering problem revised

If we look at the probability measure $\mathbb{P}$ as being absolutely continuous with respect to a new probability measure $\mathbb{Q}$ (the one that makes the observation process a Brownian motion)

$$d\mathbb{P} = \Lambda_t d\mathbb{Q} \quad \text{on the sigma-algebra } \mathcal{F}_t^Y,$$

then a simple application of the Bayes' formula for conditional expectations gives

$$\pi_t(\varphi) = \mathbb{E}[\varphi(X_t)|\mathcal{F}_t^Y] = \frac{\mathbb{E}_{\mathbb{Q}}[\varphi(X_t)\Lambda_t|\mathcal{F}_t^Y]}{\mathbb{E}_{\mathbb{Q}}[\Lambda_t|\mathcal{F}_t^Y]} = \frac{\sigma_t(\varphi)}{\sigma_t(1)},$$

where we denoted

$$\sigma_t(\varphi) := \int_{\mathbb{R}^d} \varphi(x)\sigma_t(dx) := \mathbb{E}_{\mathbb{Q}}[\varphi(X_t)\Lambda_t|\mathcal{F}_t^Y].$$

The advantage of switching from $\{\pi_t\}_{t\in[0,T]}$ to $\{\sigma_t\}_{t\in[0,T]}$ is clear from the following.

Theorem (Zakai 1969)

For any $\varphi \in C_b^2(\mathbb{R}^d)$ the stochastic process $\{\sigma_t(\varphi)\}_{t\in[0,T]}$ solves

$$\begin{cases} d\sigma_t(\varphi) = \sigma_t(\mathcal{L}_x\varphi)dt + \sigma_t(h\varphi)dY_t; \\ \sigma_0(\varphi) = \mathbb{E}[\varphi(X_0)]. \end{cases}$$

The filtering problem revised

Assuming uniform ellipticity for the second order part of $\mathcal{L}_x$ and existence of a density for the initial data X_0, we get the absolute continuity of σ_t with respect to the d-dimensional Lebesgue measure and we can write

$$\sigma_t(\varphi) = \int_{\mathbb{R}^d} \varphi(x)\sigma_t(dx) = \int_{\mathbb{R}^d} \varphi(x)u(t,x)dx,$$

for some $\{u(t,x)\}_{t\in[0,T],x\in\mathbb{R}^d}$

Theorem (Krylov-Rozovskii 1978, Pardoux 1979, Kunita 1982)

Under suitable regularity conditions on the coefficients of $\mathcal{L}_x^\star$, h and u_0, the stochastic partial differential equation

$$\begin{cases} du(t,x) = \mathcal{L}_x^\star u(t,x)dt + h(x)u(t,x)dY_t; \\ u(0,x) = u_0(x), \end{cases}$$

has a unique solution.

Note that, although the stochastic part of the previous equation is driven by the observation process $\{Y_t\}_{t\in[0,T]}$, such process behaves like a one dimensional Brownian motion under the probability measure $\mathbb{Q}$.

Returning to our general problem

Consider the stochastic PDE

$$\begin{cases} du(t,x,\omega) = \mathcal{A}_x u(t,x,\omega)dt + h(x)u(t,x,\omega)dB_t(\omega), & t \in [0,T], x \in \mathbb{R}^d; \\ u(0,x,\omega) = u_0(x), & x \in \mathbb{R}^d, \end{cases}$$

whose rigorous interpretation is

$$u(t,x,\omega) = u_0(x) + \int_0^t \mathcal{A}_x u(s,x,\omega)ds + h(x)\int_0^t u(s,x,\omega)dB_s(\omega);$$

here the last integral is interpreted in the Itô's sense.
To start our analysis we first write the previous identity in a differential form

$$\begin{cases} \partial_t u(t,x,\omega) = \mathcal{A}_x u(t,x,\omega) + (h(x)u(t,x,\omega)) \diamond \frac{dB_t(\omega)}{dt}, & t \in [0,T], x \in \mathbb{R}^d; \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases}$$

Here, $\frac{dB_t(\omega)}{dt}$ is the so-called Gaussian white noise (a generalized stochastic process) and $\diamond$ stands for the Wick product between $h(x)u(t,x,\omega)$ and $\frac{dB_t(\omega)}{dt}$.

We want to reduce the complexity of the previous problem by considering a one dimensional projection of the infinite dimensional object $\frac{dB_t(\omega)}{dt}$.

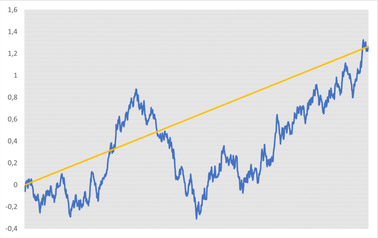
We employ a Wong-Zakai approximation argument:

- we assume T to be small;
- we consider the coarsest partition of the interval $[0, T]$, i.e. $\{0, T\}$;
- we replace $\{B_t\}_{t\in[0,T]}$ with its polygonal approximation

$$\tilde{B}_t(\omega) := \frac{t}{T} B_T(\omega), \quad t \in [0, T],$$

and hence we replace $\frac{dB_t}{dt}$ with

$$\frac{d\tilde{B}_t(\omega)}{dt} = \frac{B_T(\omega)}{T}, \quad t \in [0, T].$$



The substitution

$$\frac{dB_t}{dt} \longrightarrow \frac{B_T(\omega)}{T}, \quad t \in [0, T],$$

transforms the original problem

$$\begin{cases} \partial_t u(t,x,\omega) = \mathcal{A}_x u(t,x,\omega) + (h(x)u(t,x,\omega)) \diamond \frac{dB_t(\omega)}{dt}, & t \in [0,T], x \in \mathbb{R}^d; \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases}$$

into

$$\begin{cases} \partial_t \tilde{u}(t,x,y) = \mathcal{A}_x \tilde{u}(t,x,y) + \frac{h(x)}{T} \tilde{u}(t,x,y) \diamond y, & t \in [0,T], x \in \mathbb{R}^d; \\ \tilde{u}(0,x,y) = u_0(x), & x \in \mathbb{R}^d, \end{cases}$$

where we set $y(\omega) := B_T(\omega) \sim \mathtt{N}(0, T)$.

Wick product

Consider the Hilbert space

$$L^2(\mathbb{R},\mu)=\left\{f:\mathbb{R}\to\mathbb{R} \text{ such that } \int_{\mathbb{R}}|f(y)|^2 d\mu(y):=\int_{\mathbb{R}}|f(y)|^2\frac{1}{\sqrt{2\pi T}}e^{-\frac{y^2}{2T}}dy<+\infty\right\}.$$

The family of Hermite polynomials $\{h_n\}_{n\geq 0}$ is an orthogonal basis for $L^2(\mathbb{R},\mu)$; more precisely

- h_n is an n-th order polynomial with unit leading coefficient;
- they are orthogonal with respect to the Gaussian density:

$$\int_{\mathbb{R}} h_n(y)h_m(y)\frac{1}{\sqrt{2\pi T}}e^{-\frac{y^2}{2T}}dy=T^n n!\delta_{nm}.$$

Therefore, for any $f\in L^2(\mathbb{R},\mu)$ there exists a sequence of real numbers $\{f_n\}_{n\geq 0}$ such that

$$f(y)=\sum_{n\geq 0} f_n h_n(y) \quad \text{in } L^2(\mathbb{R},\mu).$$

The Wick product between f and y (which, by the way, corresponds to $h_1(y)$) is defined as

$$f(y)\diamond y:=\sum_{n\geq 0} f_n h_{n+1}(y).$$

It is easy to see that in general the series on the right hand side of $f(y) \diamond y := \sum_{n \geq 0} f_n h_{n+1}(y)$ will not converge in $L^2(\mathbb{R}, \mu)$ anymore, thus proving the unboundedness of that operator. More precisely, using the identity

$$h_{n+1}(y) = \left(y - T\frac{d}{dy}\right) h_n(y), \quad n \geq 0, y \in \mathbb{R},$$

one can write

$$\begin{aligned} f(y) \diamond y &:= \sum_{n \geq 0} f_n h_{n+1}(y) = \sum_{n \geq 0} f_n \left(y - T\frac{d}{dy}\right) h_n(y) \\ &= \left(y - T\frac{d}{dy}\right) \sum_{n \geq 0} f_n h_n(y) = \left(y - T\frac{d}{dy}\right) f(y). \end{aligned}$$

It is important to remark that $\left(y - T\frac{d}{dy}\right) = T\frac{d}{dy}^{\star}$, i.e.

$$\int_{\mathbb{R}} \left[T\frac{d}{dy} f(y)\right] g(y) \frac{1}{\sqrt{2\pi T}} e^{-\frac{y^2}{2T}} dy = \int_{\mathbb{R}} f(y) \left[\left(y - T\frac{d}{dy}\right) g(y)\right] \frac{1}{\sqrt{2\pi T}} e^{-\frac{y^2}{2T}} dy.$$

This tells that the operator $f(y) \mapsto f(y) \diamond y$ is the Gaussian divergence.

Back to our problem

We started with

$$\begin{cases} \partial_t u(t,x,\omega) = \mathcal{A}_x u(t,x,\omega) + (h(x)u(t,x,\omega)) \diamond \frac{dB_t(\omega)}{dt}, & t \in [0,T], x \in \mathbb{R}^d; \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases}$$

which was transformed to

$$\begin{cases} \partial_t \tilde{u}(t,x,y) = \mathcal{A}_x \tilde{u}(t,x,y) + \frac{h(x)}{T} \tilde{u}(t,x,y) \diamond y, & t \in [0,T], x \in \mathbb{R}^d; \\ \tilde{u}(0,x,y) = u_0(x), & x \in \mathbb{R}^d. \end{cases}$$

Now, using the definition of Wick product we can rewrite the last problem as

$$\begin{cases} \partial_t \tilde{u}(t,x,y) = \mathcal{A}_x \tilde{u}(t,x,y) + \frac{h(x)}{T}(y - T\partial_y)\tilde{u}(t,x,y), & t \in [0,T], x \in \mathbb{R}^d, y \in \mathbb{R}; \\ \tilde{u}(0,x,y) = u_0(x), & x \in \mathbb{R}^d, \end{cases}$$

or equivalently

$$\begin{cases} \partial_t \tilde{u}(t,x,y) = \mathcal{A}_x \tilde{u}(t,x,y) - h(x)\partial_y \tilde{u}(t,x,y) + \frac{h(x)}{T} y \tilde{u}(t,x,y), & t \in [0,T], x \in \mathbb{R}^d, y \in \mathbb{R}; \\ \tilde{u}(0,x,y) = u_0(x), & x \in \mathbb{R}^d. \end{cases}$$

Back to our problem

Setting

$$v(t,x,y) = \tilde{u}(t,x,y)e^{-\frac{y^2}{2T}}, \quad t \in [0,T], x \in \mathbb{R}^d, y \in \mathbb{R}$$

we can reduce problem

$$\begin{cases} \partial_t \tilde{u}(t,x,y) = \mathcal{A}_x \tilde{u}(t,x,y) - h(x)\partial_z \tilde{u}(t,x,y) + \frac{h(x)}{T} y \tilde{u}(t,x,y), & t \in [0,T], x \in \mathbb{R}^d, y \in \mathbb{R}; \\ \tilde{u}(0,x,y) = u_0(x), & x \in \mathbb{R}^d, \end{cases}$$

to

$$\begin{cases} \partial_t v(t,x,y) = \mathcal{A}_x v(t,x,y) - h(x)\partial_z v(t,x,y), & t \in [0,T], x \in \mathbb{R}^d, y \in \mathbb{R}; \\ v(0,x,y) = u_0(x)e^{-\frac{y^2}{2T}}, & x \in \mathbb{R}^d, y \in \mathbb{R}. \end{cases}$$

Main result

Theorem

Let v be a classical solution of the Cauchy problem

$$\begin{cases} \partial_t v(t,x,y) = \mathcal{L}_x^\star v(t,x,y) - h(x)\partial_y v(t,x,y), & t \in [0,T], x \in \mathbb{R}^d, y \in \mathbb{R}; \\ v(0,x,y) = u_0(x)e^{-\frac{y^2}{2T}}, & x \in \mathbb{R}^d, y \in \mathbb{R}. \end{cases}$$

If $\{u(t,x)\}_{t\in[0,T],x\in\mathbb{R}^d}$ denotes the solution to the Zakai equation

$$\begin{cases} du(t,x) = \mathcal{L}_x^\star u(t,x)dt + h(x)u(t,x)dY_t, & t \in [0,T], x \in \mathbb{R}^d; \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases}$$

then for any $q \geq 1$ and $K > 0$ we have that

$$\sup_{|x|\leq K} \hat{\mathbb{E}}\left[\left|u(T,x) - e^{\frac{(Y_T - y_0)^2}{2T}} v(T,x,Y_T - y_0)\right|^q\right]^{1/q} \leq \mathcal{C}T.$$

Some comments

If we rewrite the Zakai equation in the Stratonovich form

$$\begin{cases} du(t,x) = \left[\mathcal{L}_x^\star u(t,x) - \frac{1}{2}h^2(x)u(t,x)\right] dt + h(x)u(t,x) \circ dY_t, & t \in [0,T], x \in \mathbb{R}^d; \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases}$$

and we replace as before $\{B_t\}_{t\in[0,T]}$ with its polygonal approximation

$$\tilde{B}_t(\omega) := \frac{t}{T} B_T(\omega), \quad t \in [0,T],$$

we end up with

$$\begin{cases} \partial_t \hat{u}(t,x,y) = \mathcal{A}_x \hat{u}(t,x,y) - \frac{1}{2}h^2(x)\hat{u}(t,x,y) + \frac{h(x)}{T} y \hat{u}(t,x,y), & t \in [0,T], x \in \mathbb{R}^d, y \in \mathbb{R}; \\ \hat{u}(0,x,y) = u_0(x), & x \in \mathbb{R}^d. \end{cases}$$

Hu-Kallianpur-Xiong (2002) proved that

$$\sup_{|x|\leq K} \hat{\mathbb{E}}\left[|u(T,x) - \hat{u}(T,x,Y_T - y_0)|^q\right]^{1/q} \leq \mathcal{C}\sqrt{T},$$

i.e. slower rate of convergence.

Our main result: sketch of the proof

We start with some notation:

- the generator $\mathcal{L}_x$ of the signal process $X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t m(X_s)dB_s$ is

$$\mathcal{L}_x\varphi(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\partial^2_{x_ix_j}\varphi(x) + \sum_{i=1}^{d} b_i(x)\partial_{x_i}\varphi(x),$$

  where

$$a_{ij}(x) := \sum_{k=1}^{d} m_{ik}(x)m_{jk}(x);$$

- the adjoint operator $\mathcal{L}_x^\star$ is given by

$$\mathcal{L}_x^\star\varphi(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\partial^2_{x_ix_j}\varphi(x) + \sum_{i=1}^{d} b_i^\star(x)\partial_{x_i}\varphi(x) + c(x)\varphi(x),$$

  with

$$b_i^\star(x) := \sum_{j=1}^{d}\partial_{x_j}a_{ij}(x) - b_i(x) \quad \text{and} \quad c(x) := \sum_{i=1}^{d}\left(\frac{1}{2}\sum_{j,k=1}^{d}\partial^2_{x_jx_k}a_{ij}(x) - \partial_{x_k}b_i(x)\right).$$

Our main result: sketch of the proof

It is convenient to split the operator $\mathcal{L}_x^\star$ as

$$\mathcal{L}_x^\star \varphi(x) = \mathrm{L}_x^\star \varphi(x) + c(x)\varphi(x).$$

With this notation at hand, our Cauchy problem takes the form

$$\begin{cases} \partial_t v(t,x,y) = \mathrm{L}_x^\star v(t,x,y) + c(x)v(t,x,y) - h(x)\partial_y v(t,x,y) \\ \quad (t,x,y) \in]0,T] \times \mathbb{R}^d \times \mathbb{R}; \\ v(0,x,y) = u_0(x) e^{-\frac{y^2}{2T}}, \quad (x,y) \in \mathbb{R}^d \times \mathbb{R}. \end{cases}$$

By the Feynman-Kac formula,

$$\begin{aligned} v(T,x,y) &= \hat{\mathbb{E}}\left[u_0(\hat{\xi}_T^x) e^{-\frac{\left(y - \int_0^T h(\hat{\xi}_s^x)ds\right)^2}{2T}} e^{\int_0^T c(\hat{\xi}_s^x)ds} \right] \\ &= e^{-\frac{y^2}{2T}} \hat{\mathbb{E}}\left[u_0(\hat{\xi}_T^x) e^{\int_0^T c(\hat{\xi}_s^x)ds} e^{\frac{y\int_0^T h(\hat{\xi}_s^x)ds}{T} - \frac{\left(\int_0^T h(\hat{\xi}_s^x)ds\right)^2}{2T}} \right], \end{aligned}$$

where $\{\hat{\xi}_s^x\}_{s\in[0,T]}$ solves the SDE

$$d\hat{\xi}_s^x = b^\star(\hat{\xi}_s^x) + \sigma(\hat{\xi}_s^x) d\hat{B}_s, \quad \hat{\xi}_0^x = x,$$

defined on an auxiliary probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$.

Our main result: sketch of the proof

This gives

$$e^{\frac{(Y_T-y_0)^2}{2T}} v(T,x,Y_T-y_0) = \hat{\mathbb{E}}\left[u_0(\hat{\xi}_T^x)e^{\int_0^T c(\hat{\xi}_s^x)ds} e^{\frac{(Y_T-y_0)\int_0^T h(\hat{\xi}_s^x)ds}{T} - \frac{\left(\int_0^T h(\hat{\xi}_s^x)ds\right)^2}{2T}}\right].$$

It is well known that the solution $u(t,x)$ to the Zakai equation also possesses a Feynman-Kac representation: see Kunita (1982). Here, we use instead an equivalent but simpler formulation due to Benth-Deck-Potthoff-Vage (1998), namely

$$u(T,x) = \hat{\mathbb{E}}\left[u_0(\hat{\xi}_T^x)e^{\int_0^T c(\hat{\xi}_s^x)ds} e^{\int_0^T h(\hat{\xi}_{T-s}^x)dY_s - \frac{1}{2}\int_0^T h^2(\hat{\xi}_s^x)ds}\right].$$

A comparison between the last two expressions gives

$$\begin{aligned}
&u(T,x) - e^{\frac{(Y_T-y_0)^2}{2T}} v(T,x,Y_T-y_0)\\
&\quad = \hat{\mathbb{E}}\left[u_0(\hat{\xi}_T^x)e^{\int_0^T c(\hat{\xi}_s^x)ds} e^{\int_0^T h(\hat{\xi}_{T-s}^x)dY_s - \frac{1}{2}\int_0^T h^2(\hat{\xi}_s^x)ds}\right]\\
&\qquad - \hat{\mathbb{E}}\left[u_0(\hat{\xi}_T^x)e^{\int_0^T c(\hat{\xi}_s^x)ds} e^{\frac{(Y_T-y_0)\int_0^T h(\hat{\xi}_s^x)ds}{T} - \frac{\left(\int_0^T h(\hat{\xi}_s^x)ds\right)^2}{2T}}\right]\\
&\quad = \hat{\mathbb{E}}\left[u_0(\hat{\xi}_T^x)e^{\int_0^T c(\hat{\xi}_s^x)ds}\left(e^{\int_0^T h(\hat{\xi}_{T-s}^x)dY_s - \frac{1}{2}\int_0^T h^2(\hat{\xi}_s^x)ds} - e^{\frac{(Y_T-y_0)\int_0^T h(\hat{\xi}_s^x)ds}{T} - \frac{\left(\int_0^T h(\hat{\xi}_s^x)ds\right)^2}{2T}}\right)\right].
\end{aligned}$$

Then,

$$|u(T,x) - e^{\frac{(Y_T - y_0)^2}{2T}} v(T,x,Y_T - y_0)|$$
$$\leq \hat{\mathbb{E}}\left[|u_0(\hat{\xi}^x_T)| e^{\int_0^T c(\hat{\xi}^x_s)ds} \left| e^{\int_0^T h(\hat{\xi}^x_{T-s})dY_s - \frac{1}{2}\int_0^T h^2(\hat{\xi}^x_s)ds} - e^{\frac{(Y_T - y_0)\int_0^T h(\hat{\xi}^x_s)ds}{T} - \frac{\left(\int_0^T h(\hat{\xi}^x_s)ds\right)^2}{2T}} \right| \right];$$

the boundedness of u_0 and h reduce the problem to estimate of

$$\hat{\mathbb{E}}\left[\left| e^{\int_0^T h(\hat{\xi}^x_{T-s})dY_s - \frac{1}{2}\int_0^T h^2(\hat{\xi}^x_s)ds} - e^{\frac{(Y_T - y_0)\int_0^T h(\hat{\xi}^x_s)ds}{T} - \frac{\left(\int_0^T h(\hat{\xi}^x_s)ds\right)^2}{2T}} \right| \right].$$

It is crucial to work with respect to the probability measure $\hat{\mathbb{P}}$ which makes $Y_t - y_0$ a one dimensional Brownian motion.