Asymptotic expansion of the nonlocal heat content

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based on the joint work with Tomasz Grzywny

Classical perimeter

De Giorgi proposed the following definition of a perimeter of a measurable set $\Omega \subset \mathbb{R}^d$, based on the heat semigroup $(T_t)_{t\geq 0}$ in $\mathbb{R}^d$:

$$\mathrm{Per}(\Omega) := \lim_{t\to 0} \|\nabla_x T_t \mathbb{1}_\Omega\|_{L^1(\mathbb{R}^d)}.$$

Here

$$T_t u(x) = \int_{\mathbb{R}^d} g_t(x,y) u(y)\,\mathrm{d}y$$

for every $u \in L^1(\mathbb{R}^d)$, where

$$g_t(x,y) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x-y|^2}{4t}}.$$

Classical heat content

The heat content of a Borel measurable set $\Omega \subset \mathbb{R}^d$ at time t is defined as

$$\mathbb{H}_\Omega(t) = \int_\Omega T_t \mathbb{1}_\Omega(x)\, \mathrm{d}x,$$

with $(T_t)_{t \geq 0}$ being the heat semigroup in $L^2(\mathbb{R}^d)$.

Note that $u(t,x) = \int_\Omega g_t(x,y)\mathrm{d}y$ is the weak solution to the initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta\, u(t,x), & (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ u(0,x) = \mathbb{1}_\Omega(x), & x \in \mathbb{R}^d. \end{cases}$$

The heat content represents the amount of heat in Ω at time t if the initial temperature in Ω is 1 and the initial temperature in Ω^c is 0.

Asymptotics

Theorem (van den Berg, Gilkey, 1994)

For an open set $\Omega \subset \mathbb{R}^d$, $d \geq 2$, *of finite Lebesgue measure* $|\Omega|$ *and smooth* $\mathcal{C}^\infty$ *boundary,*

$$\mathbb{H}_\Omega(t) \simeq \sum_{k=0}^{\infty} \beta_k t^{k/2},$$

as $t \to 0^+$, *where*

$$\beta_0 = |\Omega|,$$

$$\beta_1 = -\frac{2}{\sqrt{\pi}} \operatorname{Per}(\Omega),$$

$$\beta_2 = \frac{d-1}{2} \int_{\partial\Omega} H(s)\, \mathrm{d}s,$$

and β_3, β_4 *have an explicit form. Here* $H(s)$ *denotes mean curvature of the set* $\partial\Omega$ *at a point* $s \in \partial\Omega$.

Convolution semigroups

Let $d \in \mathbb{N}$ and let p_t be a probabilistic convolution semigroup defined by

$$\int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle} p_t(\mathrm{d}x) = e^{-t\psi(\xi)}\,, \qquad \xi \in \mathbb{R}^d$$

where

$$\psi(x) = \int_{\mathbb{R}^d} \left(1 - e^{i\langle x, z\rangle}\right) \nu(\mathrm{d}z)\,,$$

and $\nu(\mathrm{d}z)$ is a Borel measure satisfying

$$\nu(\{0\}) = 0\,, \qquad \int_{\mathbb{R}^d} (1 \wedge |z|)\, \nu(\mathrm{d}z) < \infty\,.$$

Convolution semigroups

Let

$$P_t f(x) = \int_{\mathbb{R}^d} f(x+y)\, p_t(\mathrm{d}y).$$

The generator of the semigroup $(P_t)_{t\geq 0}$ is defined as

$$\mathcal{L}f(x) = \lim_{t\to 0^+} \frac{P_t f(x) - f(x)}{t}.$$

For Lipschitz functions we have

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d} (f(x+y) - f(x))\, \nu(\mathrm{d}y).$$

Convolution semigroups

We consider radial, continuous and non-decreasing majorant of $\mathrm{Re}\,\psi$, defined by

$$\psi^*(r) = \sup_{|z|\leq r} \mathrm{Re}[\psi(z)], \qquad r > 0\,.$$

For $r > 0$ we define the *concentration function*

$$h(r) = \int_{\mathbb{R}^d} \left(1 \wedge \frac{|x|^2}{r^2}\right) \nu(\mathrm{d}x)\,.$$

We have

$$\frac{1}{8(1+2d)} h(1/r) \leq \psi^*(r) \leq 2h(1/r)\,, \qquad r > 0\,.$$

Thus h is a more tractable version of ψ^*.

Upper scaling condition

Let $\theta_0 \geq 0$ and $\phi : (\theta_0, \infty) \to [0, \infty]$. We say that ϕ satisfies the *weak upper scaling condition* (at infinity) if there are numbers $\alpha \in \mathbb{R}$, and $\overline{C} \in [1, \infty)$ such that

$$\phi(\lambda\theta) \leq \overline{C}\lambda^{\alpha}\phi(\theta) \quad \text{for} \quad \lambda \geq 1, \quad \theta > \theta_0.$$

In short, $\phi \in \mathrm{WUSC}\left(\alpha, \theta_0, \overline{C}\right)$.

Hölder space

We will consider the Hölder space

$$\mathcal{C}_0^\beta = \left\{ f \in C_0(\mathbb{R}^d) : \|f\|_\beta := \sup_{|x-y|\leq 1} \frac{|f(x)-f(y)|}{|x-y|^\beta} + \|f\|_\infty < \infty \right\}.$$

Nonlocal heat content

Let Ω a non-empty open subset of $\mathbb{R}^d$ with finite Lebesgue measure $|\Omega|$. We consider the following quantity associated with the semigroup p_t,

$$H_\Omega(t) = \int_\Omega \int_{\Omega - x} p_t(\mathrm{d}y)\mathrm{d}x,$$

which we will call heat content.

Nonlocal heat content

Note that $u(t,x) = \int_{\Omega - x} p_t(\mathrm{d}y)$ is the unique weak solution to the initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} = \mathcal{L}\, u(t,x), & (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ u(0,x) = \mathbb{1}_\Omega(x), & x \in \mathbb{R}^d. \end{cases}$$

Example (Fractional heat content)

Let $\alpha \in (0,2)$. The heat content related to p_t satisfying

$$\psi(x) = |x|^\alpha,$$

i.e. $\mathcal{L} = -(-\Delta)^{\alpha/2}$, is called the fractional heat content.

Nonlocal perimeter

The perimeter $\mathrm{Per}_\nu(\Omega)$ related to the measure ν is defined as

$$\mathrm{Per}_\nu(\Omega) = \int_\Omega \int_{\Omega^c - x} \nu(\mathrm{d}y)\,\mathrm{d}x.$$

For any $\Omega \subset \mathbb{R}^d$, if $|\Omega| < \infty$ and $\mathrm{Per}(\Omega) < \infty$, then $\mathrm{Per}_\nu(\Omega) < \infty$.

Example (Fractional perimeter)

$$\mathrm{Per}_{(\alpha)}(\Omega) = \int_\Omega \int_{\Omega^c} \frac{1}{|x-y|^{d+\alpha}}\,\mathrm{d}x\mathrm{d}y.$$

Covariance function

For any $\Omega \subset \mathbb{R}^d$ with finite Lebesgue measure $|\Omega|$ we define the covariance function g_Ω of Ω as follows

$$g_\Omega(y) = |\Omega \cap (\Omega + y)| = \int_{\mathbb{R}^d} \mathbb{1}_\Omega(x)\, \mathbb{1}_\Omega(x-y)\, \mathrm{d}x, \quad y \in \mathbb{R}^d.$$

By Cygan and Grzywny,

$$H_\Omega(t) = P_t g_\Omega(0).$$

Asymptotic behavior of the fractional heat content

Theorem (Acuña Valverde, 2017)

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$. *Let* $\nu^{(\alpha)}(\mathrm{d}z) = \mathcal{A}_{d,-\alpha}|z|^{-d-\alpha}\,\mathrm{d}z$.

(*i*) *For* $1 < \alpha < 2$,

$$H_\Omega(t) = |\Omega| - \frac{1}{\pi}\Gamma\left(1 - \frac{1}{\alpha}\right)\mathrm{Per}(\Omega)t^{1/\alpha} + o(t^{1/\alpha}) \quad \text{as } t \to 0^+.$$

(*ii*) *For* $\alpha = 1$,

$$H_\Omega(t) = |\Omega| - \frac{1}{\pi}\mathrm{Per}(\Omega)t\ln\left(\frac{1}{t}\right) + o\left(t\ln\left(\frac{1}{t}\right)\right) \quad \text{as } t \to 0^+.$$

(*iii*) *For* $0 < \alpha < 1$,

$$H_\Omega(t) = |\Omega| - t\,\mathrm{Per}_{\nu^{(\alpha)}}(\Omega) + o(t) \quad \text{as } t \to 0^+.$$

Asymptotic behavior of the nonlocal heat content

Theorem (Cygan, Grzywny, 2017)

Let $\Omega \subset \mathbb{R}^d$ *be an open set such that* $|\Omega| < \infty$ *and* $\mathrm{Per}(\Omega) < \infty$. *Then*

$$H_\Omega(t) = |\Omega| - t\,\mathrm{Per}_\nu(\Omega) + o(t) \quad \text{as } t \to 0^+.$$

Asymptotic expansion of $P_t f$ for $f \in \mathcal{C}_0^\beta$

Theorem 1 (TG, JL, 2022)

Assume that $\psi^* \in \mathrm{WUSC}\left(\alpha, 1, \overline{C}\right)$ *for some* $\alpha \in (0,1)$. *If* $f \in \mathcal{C}_0^\beta$ *for some* $\beta \in (n\alpha, 1]$, *then*

$$P_t f = \sum_{k=0}^{n} \frac{t^k}{k!} \mathcal{L}^k f + o(t^n) \quad \text{as } t \to 0^+.$$

Asymptotic expansion of the nonlocal heat content

Theorem 2 (TG, JL, 2022)

Assume that there exists $C > 0$ *such that for all* $\lambda \leq 1$ *and* $r < 1$, $h(\lambda r) \leq C\lambda^{-\alpha} h(r)$ *for some* $\alpha \in (0,1)$. *Let* $n \geq 2$. *If* $n\alpha < 1$, *then*

$$H_\Omega(t) = |\Omega| - t\,\mathrm{Per}_\nu(\Omega) - \sum_{k=2}^{n} \frac{t^k}{k!} \mathcal{L}^k g_\Omega(0) + o(t^n) \quad \text{as } t \to 0^+.$$

Asymptotic expansion of the fractional heat content

Example

For $\alpha \in (0,1)$, $n\alpha < 1$,

$$H_\Omega(t) = |\Omega| + \sum_{k=1}^{n} \frac{(-1)^k t^k}{k!} \operatorname{Per}_{\nu^{(k\alpha)}}(\Omega) + o(t^n) \quad \text{as } t \to 0^+.$$

Full asymptotic expansion of $P_t f$ for $f \in \mathcal{C}_0^\beta$

Theorem 3 (TG, JL, 2022)

Assume that for all $\alpha \in (0,1)$, $\psi^* \in \mathrm{WUSC}\left(\alpha, 1, C\alpha^{-a}\right)$ *for some* $C > 0$ *and* $a \in (0,1)$. *If* $f \in \mathcal{C}_0^\beta$ *for some* $\beta \in (0,1]$, *then*

$$P_t f = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathcal{L}^k f.$$

Full asymptotic expansion of the nonlocal heat content

Theorem 4 (TG, JL, 2022)

Assume that for all $\alpha \in (0,1)$ *and* $\lambda \leq 1$, $h(\lambda r) \leq C\alpha^{-a}\lambda^{-\alpha}h(r)$ *for some* $C > 0$ *and* $a \in (0,1)$. *Then,*

$$H_\Omega(t) = |\Omega| - t\,\mathrm{Per}_\nu(\Omega) + \sum_{k=2}^{\infty} \frac{t^k}{k!}\mathcal{L}^k g_\Omega(0).$$

Example

Example

Let

$$\nu(\mathrm{d}x) = (x(1+\log(1+1/x)))^{-1}\mathbb{1}_{(0,2)}(x)\,\mathrm{d}x.$$

Then

$$h(u) \sim \log(1+\log(1+1/u)).$$

Since for all $\alpha \in (0,1), \log(1+x) \in \mathrm{WUSC}\,(\alpha, 0, 1/\alpha)$, then for all $\alpha \in (0,1)$ and $\lambda \leq 1$, $h(\lambda r) \leq C/\sqrt{\alpha}\lambda^{-\alpha}h(r)$ for some $C > 0$. Hence h satisfies the assumptions of the previous theorem.

Asymptotic expansion of the fractional heat content

Theorem 5 (TG, JL, 2022)

Let $\alpha \in (0,1)$ *be such that* $1/\alpha \notin \mathbb{N}$. *Then*

$$H_\Omega(t) = |\Omega| + \sum_{n=1}^{\left[\frac{1}{\alpha}\right]} \frac{(-1)^n}{n!} t^n \operatorname{Per}_{\nu^{(n\alpha)}}(\Omega) + C(d,\alpha)\operatorname{Per}(\Omega) t^{1/\alpha} + o(t^{1/\alpha})$$

as $t \to 0^+$, *where*

$$C(d,\alpha) = \frac{\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)} \left(\int_0^1 r^d p_1(re_d)\,\mathrm{d}r - \sum_{n=1}^{\infty} \frac{a_n}{1-n\alpha} \right),$$

$$a_n = \frac{1}{\pi^{1+d/2}} \frac{(-1)^{n-1}}{n!} 2^{n\alpha} \Gamma\left(\frac{n\alpha}{2}+1\right) \Gamma\left(\frac{n\alpha+d}{2}\right) \sin\left(\frac{\pi n\alpha}{2}\right).$$

Asymptotic expansion of the fractional heat content

Theorem 6 (TG, JL, 2022)

Let $\alpha \in (0,1)$ *be such that* $1/\alpha \in \mathbb{N}$. *Then*

$$H_\Omega(t) = |\Omega| + \sum_{n=1}^{1/\alpha - 1} \frac{(-1)^n}{n!} t^n \operatorname{Per}_{\nu^{(n\alpha)}}(\Omega) + \frac{(-1)^{1/\alpha}}{(1/\alpha - 1)!\pi} \operatorname{Per}(\Omega) t^{1/\alpha} \log(1/t) + o(t^{1/\alpha} \log(1/t))$$

as $t \to 0^+$.

Bibliography

L. Acuña Valverde.
Heat content for stable processes in domains of $\mathbb{R}^d$.
J. Geom. Anal., 27(1):492–524, 2017.

W. Cygan and T. Grzywny.
Heat content for convolution semigroups.
J. Math. Anal. Appl., 446(2):1393–1414, 2017.

T. Grzywny and J. Lenczewska.
Asymptotic expansion of the nonlocal heat content.
arXiv, 2022.

M. van den Berg and P. B. Gilkey.
Heat content asymptotics of a Riemannian manifold with boundary.
J.Funct. Anal., 120(1):48–71, 1994.