A doubly nonlinear evolutionary PDE

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Outline

The talk in short

The equation

$$\left|\frac{\partial v}{\partial t}\right|^{p-2}\frac{\partial v}{\partial t} = \Delta_p v := \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right), \quad p \geq 2$$

- Large time behavior and connection to the eigenvalue problem

$$\lambda_p = \inf_{u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx}, \quad \Omega \text{ bounded}$$

- Regularity

The heat equation

If

$$\begin{cases} v_t = \Delta v, & x \in \Omega \\ v(x,t) = 0, & x \in \partial\Omega \\ v(x,0) = g(x), & x \in \Omega \end{cases}$$

then (under some assumptions)

$$e^{\lambda t} v \to u, \quad \lambda = \inf_{u \in W_0^{1,2}(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}$$

where u is a ground state, i.e.

$$\begin{cases} \Delta u = -\lambda u, & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases}$$

The heat equation cont'd

The reason: We have the eigenfunction expansion:

$$v = \sum_{1}^{\infty} a_k(t) u_k e^{-\lambda_k t},$$

where u_k is the kth (normalized) eigenfunction, λ_k the kth eigenvalue and

$$a_k = \int_\Omega g u_k \, dx.$$

An alternative nonlinear argument

One can also determine the large time behavior by using that the quantities

$$\int_\Omega \left| e^{\lambda t} \nabla v \right|^2 dx, \quad \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega |v|^2 dx}$$

are non-increasing in time.

Our motivation

Find a "good evolutionary equation" related to functional inequalities when we do not have the Hilbert space structure and understand the connection with the large time behavior, in the sense that

$$v(x,t) = f(t)u + \text{higher order terms}$$

where $f(t) \to 0$ and u is a ground state (extremal), for a solution $v(x,t)$. In the best case scenario, also control the second term.

The p-eigenvalue problem

The Rayleigh quotient in $W^{1,p}$:

$$\lambda_p = \inf_{u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx}$$

The eigenvalue equation for the extremal (ground state):

$$\begin{cases} \Delta_p u + \lambda_p |u|^{p-2} u = 0, & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases}$$

Known result: The p-eigenvalue problem

- Minimizers equivalent to solutions and the equation admits a non-negative solution iff $\lambda = \lambda_p$.
- The extremal is unique up to a multiplicative constant (Ω connected). Thelin (balls), Sakaguchi (convex domains), Anane ($C^{2,\alpha}$-domains), Lindqvist (any bounded domain).
- The first eigenvalue is isolated.
- Any higher (sign-changing) eigenfunction has a finite number of nodal domains (connected components of $\{\pm u > 0\}$.
- The restriction of an eigenfunction to one of its nodal domains is a first eigenfunction in the nodal domain.
- **Unknown** if the eigenvalues are countable ($p \neq 2$) and if they form basis.

The evolution equation

We found that the equation

$$|v_t|^{p-2}v_t = \Delta_p v$$

seemed to have some good properties.

The equation is mentioned in a footnote in the paper "On the Dirichlet boundary value problem for a degenerate parabolic equation", 1996, by Kilpeläinen and Lindqvist. They call it the "artificial equation".

The evolution equation cont'd

Observations:

- The equation is invariant under multiplication with constants.
- The function $v = e^{-\lambda_p^{\frac{1}{p-1}} t} u$, with u a ground state solves the equation.
- The Rayleigh quotient $\dfrac{\displaystyle\int_\Omega |\nabla v|^p dx}{\displaystyle\int_\Omega |v|^p dx}$ decreases along solutions.

Further comments on the equation

- Uniqueness of weak solutions not known. Standard methods do not apply.
- Uniqueness of viscosity solutions. Follows from standard arguments.
- No standard regularity theory available. In contrast to the situation for Trudinger's equation

$$|v|^{p-2}v_t = \Delta_p v$$

- More suitable for viscosity solutions than for weak solutions.
- The equation degenerates when the time derivative *or* the gradient is small.

Notion of solutions

Weak: We define weak solutions as functions v with $v \in L^\infty(\mathbb{R}^+; W_0^{1,p}(\Omega))$ and $v_t \in L^p(\mathbb{R}^+ \times \Omega)$ satisfying

$$\int_\Omega |v_t|^{p-2} v_t \phi \, dx + \int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla \phi \, dx = 0$$

for a.e. $t > 0$, for each $\phi \in W_0^{1,p}(\Omega)$.

Viscosity: As usual.

Results

There is a weak solution which is also the unique viscosity solution.

Theorem: The limit (in $W^{1,p}$)

$$w(x) = \lim_{t\to\infty} e^{\mu_p t} v(x,t), \quad \mu_p := \lambda_p^{\frac{1}{p-1}}$$

is a ground state if it is not zero. In particular for strictly positive initial data.

Related results

- Similar results for $w^{p-2}w_t = \Delta_p w$, *Trudinger's equation*. Stan-Vásquez 2013, Hynd-L., 2021.
- Similar results for general p-homogenous norms in Banach spaces. Hynd-L., 2019. Also generalized by Bungert-Burger, 2020.
- Large time behavior for the doubly nonlinear p-porous medium type equations: Agueh, Aronson, Blanchet, Bonforte, Carillo, Del Pino, Dolbeault, Grillo, Manfredi, Vespri, Peletier, Stan, Vázquez.

Monotonicity

Monotonicity:

Let $\mu_p = \lambda_p^{\frac{1}{p-1}}$. Then

$$\int_\Omega \left| e^{\mu_p t} \nabla v \right|^p dx \quad \text{and} \quad \frac{\int_\Omega |\nabla v|^p dx}{\int_\Omega |v|^p dx}$$

are nonincreasing in t.

Proof monotonicity 1

With v as a test function and Hölder

$$\begin{aligned}\int_\Omega |\nabla v|^p dx &= \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla v dx \\ &= -\int_\Omega |v_t|^{p-2} v_t \cdot v dx \\ &\leq \left(\int_\Omega |v_t|^p dx\right)^{1-1/p} \left(\int_\Omega |v|^p dx\right)^{1/p}\end{aligned}$$

By the definition of λ_p, this implies

$$\int_\Omega |v_t|^p dx \geq \lambda_p^{\frac{1}{p-1}} \int_\Omega |\nabla v|^p dx$$

Proof monotonicity 1

The first monotonicity now follows from

$$\frac{d}{dt}\int_\Omega \frac{1}{p}|\nabla v|^p = \int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla v_t = -\int_\Omega |v_t|^p \leq -\mu_p\int_\Omega |\nabla v|^p$$

Proof for the Rayleigh quotient: 1

We have

$$\frac{d}{dt}\frac{\int_\Omega |\nabla v|^p}{\int_\Omega |v|^p} = -p\frac{\int_\Omega |v_t|^p}{\int_\Omega |v|^p} - p\frac{\int_\Omega |\nabla v|^p}{\left(\int_\Omega |v|^p\right)^2}\int_\Omega |v|^{p-2}vv_t$$

$$= \frac{p}{\left(\int_\Omega |v|^p\right)^2}\left\{\int_\Omega |\nabla v|^p \int_\Omega |v|^{p-2}v(-v_t) - \int_\Omega |v|^p \int_\Omega |v_t|^p\right\}$$

Proof for the Rayleigh quotient: 2

From Hölder's inequality

$$\int_\Omega |v|^{p-2} v(-v_t)dx \leq \left(\int_\Omega |v|^p dx\right)^{1-1/p} \left(\int_\Omega |v_t|^p dx\right)^{1/p}$$

Together with Step 1

$$\int_\Omega |\nabla v|^p dx \int_\Omega |v|^{p-2} v(-v_t dx) \leq \int_\Omega |v_t|^p dx \int_\Omega |v|^p dx$$

Thus

$$\frac{d}{dt} \frac{\int_\Omega |\nabla v|^p dx}{\int_\Omega |v|^p dx} \leq 0$$

Proof of large time asymptotics 1:

Let

$$u(x,t) := e^{\mu_p t} v(x,t), \quad \mu_p = \lambda_p^{\frac{1}{p-1}}$$

By the previous monotonicity, we may consider the limit

$$S := \lim_{t\to\infty} \int_\Omega |\nabla u|^p dx.$$

Proof of large time asymptotics 2:

For $s_k \to \infty$ let

$$v^k(x,t) = e^{\mu_p s_k} v(x, t + s_k)$$

v^k is a solution with initial data

$$g^k(x) = e^{\mu_p s_k} v(x, s_k), \quad \text{uniformly bdd in } W_0^{1,p}(\Omega)$$

From a compactness argument $v_{k_j}(x,t) \to v(x,t)$ in $W_0^{1,p}(\Omega)$ for a.e. t, where v is a weak solution (upon extracting a subsequence).

Proof of large time asymptotics 3:

Define $w(x,t) := e^{\mu_p t} v(x,t)$, then

$$S = \lim_{j\to\infty} \int_\Omega |\nabla u^{k_j}|^p dx = \int_\Omega |\nabla w|^p dx.$$

Checking the case of equality for the monotonicity, we see that $w(\cdot,t)$ is a p-ground state for each $t \geq 0$ with constant norm S.

Proof of large time asymptotics 4:

From the simplicity of ground states it is the same ground state for each $t \geq 0$.

If we take another subsequence $s_k \to \infty$ still (from the monotonicity)

$$\int_\Omega |\nabla w|^p dx = S.$$

So the limit will be the same.

What regularity can be expected?

Stationary solutions are p-harmonic that are in general not C^2.

This can also be seen by the solution

$$Ct + |x|^{\frac{p}{p-1}}.$$

Related regularity results

- Regularity for the p-parabolic equation, $u_t = \Delta_p u$, DiBenedetto and many others: $C_x^{1,\alpha}$ and $C_t^{\frac{1}{2}}$.
- Cyril Imbert, Tianling Jin, and Luis Silvestre, 2015-2016, gradient estimates parabolic equations involving the p-Laplacian:

$$u_t = |\nabla u|^q \Delta_p u, \quad q \in (1-p, \infty)$$

$C_x^{1,\alpha}$ and $C_t^{\frac{1}{2}+\varepsilon}$.

- Imbert, Silvestre, 2013: Hölder estimates and Harnack inequality for equations that are elliptic when the gradient is large:

$$a_{ij}u_{ij} = 0, \quad \text{when} \quad |\nabla u| \geq 1.$$

Regularity result

Theorem: Suppose u is a viscosity solution in $B_2 \times (-2, 0]$. Then $u(x, t)$ is locally Lipschitz continuous in x and locally $\frac{p-1}{p}$-Hölder continuous and

$$|u(x,t)-u(y,s)| \leq C(n,p)\|u\|_{L^\infty(B_2\times(-2,0])} \left(|x-y| + |t-s|^{\frac{p-1}{p}}\right),$$

for any $(x, t),\ (y, s) \in B_1 \times (-1, 0]$.

Corollary: The convergence $e^{\mu_p t} v(x, t) \to$ a ground state is uniform (under some assumptions on g).

Comparison

For many elliptic/parabolic equations a comparison principle holds. In the viscosity setting, this may be proved by “doubling the variables”. This amounts to ruling out that

$$\sup_{x,y} \left(u(x) - v(y) - \phi(|x-y|)\right) > 0$$

when u is a subsolution, v is a supersolution, $u \leq v$ on the boundary and ϕ is appropriately chosen. For uniformly elliptic equations the choice $\phi(r) = r^2$ is typical.

Ishii-Lions method

A similar approach can give continuity estimates, first by Ishii and Lions, 1990. A spatial continuity estimate of order $\phi(r)$ for a solution u is saying that

$$\sup_{x,y,t} \left(u(x,t) - u(y,t) - \phi(|x-y|)\right) \leq 0.$$

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In order to prove this, we assume towards a contradiction that

$$\sup_{x,y,t} \left(u(x,t) - u(y,t) - \phi(|x-y|)\right) > 0.$$

In our case, $\phi(r) \approx r|\ln r|$ and $\phi(r) \approx r$. This gives a log-Lipschitz and a Lipschitz estimate. Here ϕ will be strictly concave.

How the proof works in one dimension for $p = 2$

Assume that the positive supremum of

$$u(t,x) - u(t,y) - A\phi(|x-y|) - \frac{B}{2}\left(|x|^2 + |y|^2 + t^2\right)$$

is attained at an interior point (x, y, t).

How the proof works in one dimension for $p = 2$

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is attained at an interior point (x, y, t).Then

$$\begin{aligned} u_t(t,x) - u_t(t,y) &= Bt \\ u_{xx}(t,x) - u_{yy}(t,y) &\leq 2B + 2A\phi''(|x-y|). \end{aligned}$$

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$$u_t(t,x) - u_t(t,y) = Bt$$
$$u_{xx}(t,x) - u_{yy}(t,y) \leq 2B + 2A\phi''(|x-y|).$$

The equation gives

$$Bt \leq 2B + 2A\phi''(|x-y|),$$

which is a contradiction for A large.

Time regularity

The spatial regularity can then be used to construct a suitable supersolution which yields the desired time regularity. We claim that

$$u(x,t) - u(0,t_0) \leq \psi(t,x) := \eta + A(t-t_0) + B|x|^{\frac{p}{p-1}}, \qquad (4.1)$$

This is accomplished by making ψ a supersolution and applying the comparison principle by choosing A and B in a good way so that this holds on the parabolic boundary of a cylinder.

Comments/questions

- Higher order expansion at $t = \infty$?
- Is there a natural orthogonality condition as when $p = 2$?
- Harnack inequality (in what form)?
- Higher regularity theory?
- General theory for equations that are uniformly parabolic when the gradient and the time derivative are bounded away frome zero?
- Asymptotic behaviour as $t \to \infty$ for the evolution equation for $p = \infty$?
- Can we prove any concavity/convexity properties for ground states using the evolution equation?

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Thank you for your attention!