On the Dirichlet problem for Kolmogorov-Fokker-Planck type equations with rough coefficients

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Introduction

Let U_X be a bounded Lipschitz domain and $V_{Y,t}$ be a bounded domain with $C^{1,1}$ boundary, and consider the problem

$$\begin{cases} \mathcal{L}u = g^* & \text{in } U_X \times V_{Y,t}, \\ u = g & \text{on } \partial_{\mathcal{K}}(U_X \times V_{Y,t}), \end{cases}$$

where

$$\mathcal{L} := \nabla_X \cdot (A(X,Y,t)\nabla_X) + X \cdot \nabla_Y - \partial_t,$$

with $(X,Y,t) \in \mathbb{R}^{2m+1}$, and A only measurable, bounded, symmetric and uniformly elliptic as an $m \times m$-matrix.

Background and Motivation

In a recent series of papers a number of results concerning solutions to $\mathcal{L}u = 0$ in Lipschitz type domains have been developed. These results include

- boundary Hölder estimates,
- boundary comparison principles,
- boundary Harnack inequalities,
- doubling properties of associated parabolic measures.

Furthermore, we have studied when the Radon-Nikodym derivative of these parabolic measures define an A_∞-weight w.r.t. the surface measure.

Background and Motivation

We have also established a structure theorem concerning the absolute continuity of elliptic and parabolic measures which allowed us to reprove previously established results, as well as deduce new ones, for example in the context of homogenization for Kolmogorov type operators.

Background and Motivation

Through these studies we were motivated to study the Dirichlet problem for $\mathcal{L}$ in Lipschitz type domains to gain a deeper understanding of it.

A Function Space

Define the space $W(U_X \times V_{Y,t})$ as the closure of $C^\infty(\overline{U_X \times V_{Y,t}})$ in the norm

$$||u||_{W(U_X\times V_{Y,t})} := ||u||_{L^2_{Y,t}(V_{Y,t},H^1_X(U_X))} + ||(-X\cdot\nabla_Y + \partial_t)u||_{L^2_{Y,t}(V_{Y,t},H^{-1}_X(U_X))}.$$

Additional Function Spaces

Let $N_{Y,t}$ denote the outer unit normal to $V_{Y,t}$. We define

$$\partial_{\mathcal{K}}(U_X \times V_{Y,t})$$

as

$$(\partial U_X \times V_{Y,t}) \cup \{(X,Y,t) \in \overline{U_X} \times \partial V_{Y,t} \mid (X,-1)\cdot N_{Y,t} > 0\}.$$

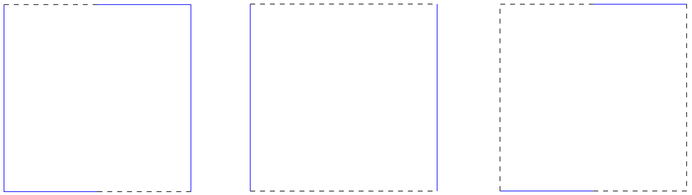
Let $C^\infty_{\mathcal{K},0}(\overline{U_X \times V_{Y,t}})$, $C^\infty_{X,0}(\overline{U_X \times V_{Y,t}})$ and $C^\infty_{Y,t,0}(\overline{U_X \times V_{Y,t}})$ consist of smooth functions which vanish on

$$\partial_{\mathcal{K}}(U_X \times V_{Y,t}), \quad \{(X,Y,t) \in \partial U_X \times \overline{V_{Y,t}}\},$$

$$\{(X,Y,t) \in \overline{U_X} \times \partial V_{Y,t} \mid (X,-1)\cdot N_{Y,t} > 0\},$$

respectively.

Additional Function Spaces



$\partial_{\mathcal{K}}(U_X \times V_{Y,t})$ $\qquad$ $\partial U_X \times \overline{V_{Y,t}}$ $\qquad$ $\{\overline{U_X} \times \partial V_{Y,t} \ : \ (X,-1) \cdot N_{Y,t} > 0\}$

We let $W_0(U_X \times V_{Y,t})$, $W_{X,0}(U_X \times V_{Y,t})$ and $W_{Y,t,0}(U_X \times V_{Y,t})$ denote the closure in the norm of $W(U_X \times V_{Y,t})$ of these spaces.

Weak Solutions

Definition

Let $g \in W(U_X \times V_{Y,t})$ and $g^* \in L^2_{Y,t}(V_{Y,t}, H_X^{-1}(U_X))$. We then say that u is a weak solution to the Dirichlet problem if

$$u \in W(U_X \times V_{Y,t}),\ (u-g) \in W_0(U_X \times V_{Y,t}),$$

and if

$$0 = \iiint_{U_X \times V_{Y,t}} A(X,Y,t)\nabla_X u \cdot \nabla_X \phi \, dXdYdt$$
$$+ \iint_{V_{Y,t}} \langle g^*(\cdot,Y,t) + (-X\cdot\nabla_Y + \partial_t)u(\cdot,Y,t), \phi(\cdot,Y,t)\rangle \, dYdt,$$

for all $\phi \in L^2_{Y,t}(V_{Y,t}, H^1_{X,0}(U_X))$ and where $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{H_X^{-1}(U_X), H^1_{X,0}(U_X)}$.

The Result

Theorem

Let U_X and $V_{Y,t}$ be domains as previously. Let A be measurable, bounded, symmetric, and uniformly elliptic. Let $g \in W(U_X \times V_{Y,t})$ and $g^* \in L^2_{Y,t}(V_{Y,t}, H_X^{-1}(U_X))$. Then there exists a unique weak solution $u \in W(U_X \times V_{Y,t})$ in the sense of the previous definition.

Furthermore, there exists a constant c, independent of u and g, but depending on m, κ, and $U_X \times V_{Y,t}$, such that

$$||u||_{W(U_X\times V_{Y,t})} \leq c\Big(||g||_{W(U_X\times V_{Y,t})} + ||g^*||_{L^2_{Y,t}(V_{Y,t},H_X^{-1}(U_X))}\Big).$$

The Kolmogorov Boundary

We defined the Kolmogorov boundary $\partial_{\mathcal{K}}(U_X \times V_{Y,t})$ as

$$(\partial U_X \times V_{Y,t}) \cup \{(X, Y, t) \in \overline{U_X} \times \partial V_{Y,t} \mid (X, -1) \cdot N_{Y,t} > 0\}.$$

Another characterization is

$$\partial_{\mathcal{K}}(U_X \times V_{Y,t}) = \bigcup_{(X,Y,t)\in U_X \times V_{Y,t}} (\mathcal{A}_{(X,Y,t)} \cap \partial(U_X \times V_{Y,t})),$$

i.e. it consists of points on the boundary that are inside the closure of the propagation set of at least one interior point.

Comment on Issues at the Boundary

The fact that we can only impose boundary data on the Kolmogorov boundary causes some issues.
In particular, close to the singular set

$$\{(X, Y, t) \in \overline{U_X} \times \partial V_{Y,t} \mid (X, -1) \cdot N_{Y,t} = 0\},$$

where propagation trajectories just graze the boundary the behaviour of the trace of a function in $W(U_X \times V_{Y,t})$ can be wild.

Our solution was to define $W_0(U_X \times V_{Y,t})$ as the closure of $C^\infty_{\mathcal{K},0}$ in the norm of $W(U_X \times V_{Y,t})$.

About the Proof

The proof was based on ideas by Brézis and Ekeland (1976), and more recent work by Armstrong and Mourrat (2019) (see also Albritton, Armstrong, Mourrat, and Novack (2021)), where the operator is associated to a convex functional on a suitable space so that null-minimizers of the functional correspond to solutions of the Dirichlet problem.

$$\textit{Solutions of the Dirichlet problem} \longleftrightarrow \textit{Null-minimizers of a functional}$$

About the Proof

Let

$$J[f, f^*] := \inf \iiint_{U_X \times V_{Y,t}} \frac{1}{2}(A(\nabla_X f - g)) \cdot (\nabla_X f - g)\, dXdYdt,$$

where
$f \in L^2_{Y,t}(V_{Y,t}, H^1_X(U_X))$, $f^* - (X \cdot \nabla_Y - \partial_t)f \in L^2_{Y,t}(V_{Y,t}, H^{-1}_X(U_X))$,
and the infimum is taken with respect to the set

$$\{g \in (L^2(V_{Y,t}, L^2(U_X)))^m \mid \nabla_X(Ag) = f^* - (X \cdot \nabla_Y - \partial_t)f\}.$$

Dirichlet problem

Then proving existence of solutions to the problem

$$\begin{cases} \mathcal{L}u = g^* & \text{in } \Omega \times \mathbb{R}^m \times \mathbb{R}, \\ u = g & \text{on } \partial\Omega \times \mathbb{R}^m \times \mathbb{R}. \end{cases},$$

where $g^* \in L^2_{Y,t}(V_{Y,t}, H^{-1}_X(U_X))$, $g \in W(U_X \times V_{Y,t})$, reduces to showing that the uniformly convex functional

$$(g + W_0) \ni f \mapsto J[f, g^*],$$

where $(g, g^*) \in W(U_X \times V_{Y,t}) \times L^2_{Y,t}(V_{Y,t}, H^{-1}_X(U_X))$, has minimum zero.

About the Proof

This is achieved through the following steps:

- First one establishes that the set $\mathcal{A}(g, g^*)$ of pairs $(f, \mathrm{j}) \in (g + W_0) \times (L^2(V_{Y,t}, L^2(U_X)))^m$ such that

$$\nabla_X \cdot (A(X, Y, t)\mathrm{j}) = g^* - (X \cdot \nabla_Y - \partial_t)f,$$

  is convex and non-empty, and that the functional

$$\mathcal{J}[f, \mathrm{j}] := \iiint_{U_X \times V_{Y,t}} \frac{1}{2}(A(\nabla_X f - \mathrm{j})) \cdot (\nabla_X f - \mathrm{j})\, dXdYdt$$

  is uniformly convex on $\mathcal{A}(g, g^*)$.

About the Proof

Then there exists a unique minimizing pair (f_1, j_1), and by construction

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- Next, one establishes the one-to-one correspondence between weak solutions to $\mathcal{L}u = g^*$ in $U_X \times V_{Y,t}$, such that $(u - g) \in W_0$, and null minimizers of $J[\cdot, g^*]$.

About the Proof

- It only remains to prove

$$\mathcal{J}[f_1, j_1] \leq 0.$$

This is achieved by studying the perturbed problem

$$G(f^*) := \inf_{f \in W_0} \left(J[f+g, f^*+g^*] - \iint_{V_{Y,t}} \langle f^*(\cdot, Y, t), f(\cdot, Y, t) \rangle \, dY dt \right),$$

and one can show that it is enough to prove that

$$G^*(h) \geq 0 \text{ for all } h \in L^2_{Y,t}(V_{Y,t}, H^1_{X,0}(U_X)),$$

where G^* is the convex dual of G.

Open Problems

Some problems to consider:

- We have defined $W(U_X \times V_{Y,t})$ as the closure of smooth functions. This gives only an implicit description and it would be interesting to find an explicit characterization.
- Traces of functions in $W(U_X \times V_{Y,t})$.

Thank you and References

Thank you for your attention!

Grazie dell'attenzione!

Thank you and References

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