Optimal regularity for degenerate Kolmogorov equations with rough coefficients

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(joint work with Andrea Pascucci and Stefano Pagliarani)

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Motivations

Stochastic differential equation in $\mathbb{R}^N$:

$$dX_t = (BX_t + b(t, X_t))\,dt + \sigma(t, X_t)\,dW_t$$

$W = (W_t)_{t \geq 0}$ Brownian motion in $\mathbb{R}^d, d \leq N$
B: real $(N \times N)$-matrix
$b : [0, T_0] \times \mathbb{R}^N \to \mathbb{R}^N, \quad b = (b_1, \ldots, b_d, 0, \ldots, 0)$
$\sigma : [0, T_0] \times \mathbb{R}^N \to \mathbb{R}^{N \times d}, \quad \sigma = \begin{pmatrix} \sigma_{d \times d} \\ 0 \end{pmatrix}$

Backward Kolmogorov operator:

$$\mathcal{L} = \underbrace{\frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^*)_{ij}\partial_{x_i x_j} + \sum_{i=1}^{d} b_i \partial_{x_i}}_{=\mathcal{A}} + \underbrace{\langle Bx, \nabla \rangle + \partial_t}_{=Y}$$

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Example from the kinetic theory

Langevin model: *Kolmogorov* (1934), *Hörmander* (1967)

$$(N = 2, d = 1) \quad \begin{cases} dV_t = b_v(t, V_t, X_t)dt + \sigma_v(t, V_t, X_t)dW_t & \text{(velocity)} \\ dX_t = V_t\, dt & \text{(position)} \end{cases}$$

Coefficients:

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} \sigma_v \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} b_v \\ 0 \end{pmatrix}$$

Kolmogorov backward operator:

$$\mathcal{L} = \underbrace{\frac{1}{2}\sigma_v^2(t, v, x)\partial_{vv} + b_v(t, v, x)\partial_v}_{=\mathcal{A}} + \underbrace{v\partial_x + \partial_t}_{=Y}$$

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Settings and assumptions

$$\mathcal{L} = \underbrace{\frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t,x)\partial_{x_i x_j} + \sum_{i=1}^{d} b_i(t,x)\partial_{x_i} + c(t,x)}_{=\mathcal{A}} + \underbrace{\langle Bx, \nabla\rangle + \partial_t}_{=Y}$$

$$(t,x) \in]0,T[\times\mathbb{R}^N =: \mathcal{S}_{T_0}$$

- **parabolic Hörmander condition**: $(d \le N)$

$$\text{rank Lie}\left(\partial_{x_1}, \dots, \partial_{x_d}, Y\right) = N+1$$

- **coercivity on** $\mathbb{R}^d$:

$$\mu^{-1}\text{Id} \le \left(a_{ij}(t,x)\right)_{i,j=1,\dots,d} \le \mu\text{Id}$$

- **coefficients**: intrinsic Hölder in space, measurable in time

$$a_{ij}, b_i, c \in L^\infty\left([0,T_0]; C^\alpha_B(\mathbb{R}^N)\right)$$

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Parabolic Hörmander condition

Equivalent statements

- the operator $\frac{\delta}{2}\Delta_d + Y$ is hypoelliptic
- B has the block form

$$B = \begin{pmatrix} * & * & \cdots & * & * \\ B_1 & * & \cdots & * & * \\ 0 & B_2 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_r & * \end{pmatrix}$$

$B_j : (d_{j-1} \times d_j)$-matrix of full rank

$$d \equiv d_0 \geq d_1 \geq \cdots \geq d_r \geq 1, \qquad \sum_{i=0}^{r} d_i = N$$

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Intrinsic geometry (Lanconelli-Polidoro, 1994)

Constant coefficients operator:

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij} \partial_{x_i x_j} + Y$$

- invariant w.r.t. left **translations**:

$$(T, y) \circ (t, x) = (T + t, e^{tB} y + x)$$

- homogeneous w.r.t. **dilations** family ($\iff$ $*$-bloks in $B = 0$)

$$D_\delta(t, x) = \mathrm{diag}(\lambda^2, \lambda I_{d_0}, \lambda^3 I_{d_1}, \dots, \lambda^{2r-1} I_{d_r})$$

$(\mathbb{R}^{N+1}, \circ, D_\delta)$ homogeneous Lie group with norm $\|(t, x)\|_B = |t|^{\frac{1}{2}} + |x|_B$

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Lie group approach

Intrinsic α-Hölder condition:

$$|f(t,x) - f(\tau,\xi)| \le \|f\|_{C_B^\alpha} \|(\tau,\xi)^{-1} \circ (t,x)\|_B^\alpha$$

- introduced by *Lanconelli-Polidoro* (1994)
- joint space and time regularity $\implies$ smoothing property
- **fundamental solution**: *Yu* interpreted as a Lie derivative

$$\mathcal{L}u = 0 \iff Yu = -\mathcal{A}u$$

  Polidoro (1995), *Di Francesco-Pascucci* (2005)
- **Schauder estimates**: *Di Francesco-Polidoro* (2006)
- **Dirichlet problem**: *Manfredini* (1997)

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Semigroup approach

Anisotropic spatial α-Hölder condition:

$$|f(t,x) - f(t,\xi)| \leq \|f\|_{C_B^\alpha} |x - \xi|_B^\alpha$$

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- **Schauder estimates**: *Lunardi* (1997), *Lorenzi* (2005), *Priola* (2009), *Biagi-Bramanti* (2022)
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Strong Lie solution with rough coefficients

Bramanti-Polidoro (2020): time-dependent measurable coefficients

$$\mathcal{L}u = 0 \iff \partial_t u = -\mathcal{A}u - \langle Bx, \nabla u \rangle \quad \text{for almost every } t$$

Pascucci-Pesce (2022): Langevin SPDE ($N = 2, d = 1$)

$$d_Y u = -\mathcal{A}u\, dt - \sigma \nabla u\, dW_t$$

$$u(\gamma_s(t,x)) - u(t,x) = -\int_t^s \mathcal{A}u(\gamma_\tau(t,x))d\tau - \int_t^s (\sigma \nabla u)(\gamma_\tau(t,x))dW_\tau$$

$s \mapsto \gamma_s(t,x)$ integral curve of Y starting from (t,x)

In our settings:

$$Yu = -\mathcal{A}u \qquad \text{a.e. along integral curves}$$

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Main results

Theorem (1.8-1.9)

Under the previous assumptions

- $\mathcal{L}$ has a fundamental solution $p = p(t, x; T, y)$ on $\mathcal{S}_{T_0}, t < T$
- Gaussian estimates: $\forall \varepsilon > 0, \quad i, j = 1, \dots, d$

$$\Gamma^{\bar{\mu}}(t, x; T, y) \lesssim p(t, x; T, y) \lesssim \Gamma^{\mu+\varepsilon}(t, x; T, y)$$

$$|\partial_{x_i} p(t, x; T, y)| \lesssim \frac{1}{\sqrt{T - t}} \Gamma^{\mu+\varepsilon}(t, x; T, y)$$

$$|\partial_{x_i x_j} p(t, x; T, y)| \lesssim \frac{1}{T - t} \Gamma^{\mu+\varepsilon}(t, x; T, y)$$

- $\forall \beta < \alpha, \tau < T$, $p(\cdot, \cdot; T, y) \in C^{2,\beta}_B(\mathcal{S}_\tau)$

$$\|p(\cdot, \cdot; T, y)\|_{C^{2,\beta}_B} \lesssim \frac{1}{(T - \tau)^{(Q+2+\beta)/2}}$$

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Intrinsic functional spaces (1)

Introduced in *Pagliarani-Pascucci-Pignotti* (2015):

Lie Hölder spaces

$$C_d^\alpha: \quad \frac{\left|f(t, x + h\,\mathbf{e}_i) - f(t,x)\right|}{|h|^\alpha} < +\infty, \quad \text{for } i = 1, \dots, d$$

$$C_Y^\alpha: \quad \frac{\left|f(\gamma_s(t,x)) - f(t,x)\right|}{|t-s|^{\frac{\alpha}{2}}} < +\infty, \quad \gamma \text{ integral curve of } Y$$

Intrinsic Hölder spaces

- $C_B^{0,\alpha}$: $f \in C_d^\alpha \cap C_Y^\alpha$
  $\implies |f(t,x) - f(\tau,\xi)| \le \|f\|_{C_B^\alpha} \|(\tau,\xi)^{-1} \circ (t,x)\|_B^\alpha$
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- $C_B^{1,\alpha}$: $f \in C_Y^{1+\alpha}, \partial_{x_i} f \in C_B^{0,\alpha}$ for $i = 1, \dots, d$

Intrinsic functional spaces (2)

Pagliarani-Pascucci-Pignotti (2015):

$$f \in C_B^{2,\alpha} \text{ if } \partial_{x_i} f \in C_B^{1,\alpha} (i = 1, \dots, d), \; Yu \in C_B^{0,\alpha}$$

Lie derivative only almost everywhere:

Intrinsic Hölder spaces of second order

$C_B^{2,\alpha}$: $\partial_{x_i} f \in C_B^{1,\alpha}$ for $i = 1, \dots, d$, $\exists f_Y \in L^\infty([0,T]; C_B^\alpha)$ such that

$$f(\gamma_s(t,x)) = f(t,x) + \int_t^s f_Y(\gamma_\tau(t,x)) d\tau$$

Intrinsic functional spaces (2)

Pagliarani-Pascucci-Pignotti (2015):

$$f \in C_B^{2,\alpha} \text{ if } \partial_{x_i} f \in C_B^{1,\alpha}(i = 1, \dots, d), \; Yu \in C_B^{0,\alpha}$$

Lie derivative only almost everywhere:

Intrinsic Hölder spaces of second order

$C_B^{2,\alpha}$: $\partial_{x_i} f \in C_B^{1,\alpha}$ for $i = 1, \dots, d$, $\exists f_Y \in L^\infty([0,T]; C_B^\alpha)$ such that

$$f(\gamma_s(t,x)) = f(t,x) + \int_t^s f_Y(\gamma_\tau(t,x)) d\tau$$

Thank you for your attention