Anomalous diffusion limit[1]

for kinetic equations with a thermostatted interface

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[1] Joint work with K. Bogdan (Wrocław University) and T. Komorowski (IMPAN)

The model: linear Boltzmann equation

We study the evolution of the phonon energy density $W(t,y,k)$ at time $t \geq 0$, as function of position $y \in \mathbb{R}$ and mode $k \in \mathbb{T} := [-1/2, 1/2]_{/\sim}$.
Outside the *interface* $\mathcal{I} := \{y = 0\}$, it reads

$$(\text{KE}) : \begin{cases} \partial_t W(t,y,k) + \omega'(k)\partial_y W(t,y,k) \; = \; \gamma L_k W(t,y,k); \\ W(0,y,k) \; = \; W_0(y,k); \end{cases}$$

where

- $\gamma > 0$ is the photon *scattering rate*;
- $W_0(y,k)$ is the initial distribution of the phonons energy;
- $\omega \in C^2(\mathbb{T}_*)$ even and unimodal, is the phonon *dispersion relation*;
- the *scattering operator* L_k, acting only on the variable k in $\mathbb{T}$, is given by

$$L_k u(k) \; := \int_{\mathbb{T}} R(k,k')\,[u(k') - u(k)]\; dk', \quad u \in B_b(\mathbb{T}), \tag{1}$$

for a *scattering kernel* $R \colon \mathbb{T} \times \mathbb{T} \to \mathbb{R}$.

The model: interface conditions

Fixed the *temperature preset* $T_o \geq 0$, let $p_+, p_-, p_0 : \mathbb{T} \to [0,1]$ be three even, continuous functions such that:

$$\textbf{[BC]:} \quad p_+(k) + p_-(k) + p_0(k) \;=\; 1.$$

Then, the density $W(t,\cdot,\cdot)$ satisfies the interface conditions:

$$(\mathrm{IC}) : \begin{cases} u(0^+,k) \;=\; p_+(k)u(0^-,k) + p_-(k)u(0^+,-k) + p_0(k)T_o, & \text{on } \mathbb{T}_+; \\ u(0^-,k) \;=\; p_+(k)u(0^+,k) + p_-(k)u(0^-,-k) + p_0(k)T_o, & \text{on } \mathbb{T}_-. \end{cases}$$

Aim: choose the right scaling index $\alpha > 0$ such that

$$W_\lambda(t,y,k) \;:=\; W(\lambda t, \lambda^{1/\alpha} y, k), \qquad \lambda > 0;$$

satisfying the interface conditions (IC) and

$$(\mathrm{rKE}): \begin{cases} \dfrac{1}{\lambda}\partial_t W_\lambda(t,y,k) + \dfrac{1}{\lambda^{1/\alpha}}\omega'(k)\partial_y W_\lambda(t,y,k) \;=\; \gamma L_k W_\lambda(t,y,k); \\ W_\lambda(0,y,k) \;=\; W_0(y,k), \end{cases}$$

admits the limit $\lim_{\lambda\to\infty} W_\lambda$ and characterize the limit function $\bar{W}$.

Physical Motivation

Chains of anharmonic oscillators (e.g. FPU-β chains) are commonly used models in non-equilibrium statistical mechanics.
⚠ hard to study the macroscopic energy transport for this non-linear dynamics.

$\rightsquigarrow$ replace the non-linearity with a random exchange of momenta between neighboring particles.

Komorowski-Olla 2020

Consider a one-dimensional infinite particle system $\{(\mathfrak{p}_y(t), \mathfrak{q}_y(t))\colon y \in \mathbb{Z}\}$ governed by the Hamiltonian:

$$\mathcal{H}(\mathfrak{p}, \mathfrak{q}) := \frac{1}{2} \sum_{y \in \mathbb{Z}} \mathfrak{p}_y^2(t) + \frac{1}{2} \sum_{y, y' \in \mathbb{Z}} \alpha_{y-y'} \mathfrak{q}_y(t) \mathfrak{q}_{y'}(t),$$

subject to a conservative stochastic exchange of momenta between adjacent particles and in contact with a Langevin thermostat at temperature T_o in $\mathcal{I}$.

$\rightsquigarrow$ The interface model (KE) + (IC) arises, in the hyperbolic scaling limit of the Wigner distribution $W^{(\epsilon)}$ of the above system.

Related results: anomalous vs classical diffusive limits

From the particle system model, the total scattering degenerates at $k = 0$:

$$R(k) := \int_{\mathbb{T}} R(k, k')\, dk' \approx |k|^2.$$

Jara-Komorowski-Olla 2009

Without the interface, the limit $\bar{W}(t, y) := \lim_\lambda W_\lambda(t, y, k)$ exists (in a weak sense), where

- Optical case: $\omega(k) \approx |k|^2 \rightsquigarrow$ diffusive scaling ($\alpha = 2$) and

$$\partial_t \bar{W}(t, y) = \bar{\gamma} \partial^2_{yy} \bar{W}(t, y);$$

- Acoustic case: $\omega(k) \approx |k| \rightsquigarrow$ super-diffusive scaling ($\alpha = 3/2$) and

$$\partial_t \bar{W}(t, y) = -\bar{\gamma} |\partial^2_{yy}|^{3/4} \bar{W}(t, y).$$

In both cases, homogenization in frequency happens at the limit:

$$\bar{W}(0, y) = \int_{\mathbb{T}} W_0(y, k)\, dk.$$

Related results: interface models

Similarly, the important feature is the behaviour of the random mechanism at $\mathcal{I}$ for $|k| \ll 1$. Assume p_+^*, p_-^*, p_0^* strictly positive, where $\phi^* := \lim_{k\to 0} \phi(k)$.

[BKO19]: In the optical case, the limit function $\bar{W}(t,y)$ satisfies

$$\begin{cases} \partial_t \bar{W}(t,y) = \bar{\gamma}\partial^2_{yy}\bar{W}(t,y), \quad y \neq 0; \\ \bar{W}(0,y) = \bar{W}_0(y), \quad \bar{W}(t,0) = T_o. \end{cases}$$

[KOR20]: In the acoustic case, the limit function $\bar{W}(t,y)$ satisfies

$$\begin{cases} \partial_t \bar{W}(t,y) = \bar{\gamma}\bar{L}_y \bar{W}(t,y), \quad y \neq 0; \\ \bar{W}(0,y) = \bar{W}_0(y); \end{cases}$$

where $q_\alpha(y) \approx |y|^{-(1+\alpha)}$ and

$$\bar{L}_y u(y) := \mathrm{p.v.} \int_{yy'>0} q_{\frac{3}{2}}(y'-y)[u(y')-u(y)]\, dy'$$

$$+ \int_{yy'<0} q_{\frac{3}{2}}(y'-y)\{p_+^*[u(y')-u(y)] + p_-^*[u(-y')-u(y)] + p_0^*[T-u(y)]\}\, dy'.$$

Our assumptions on the model

[SK]: there exist $\beta_1, \beta_2 > 0$ and even, non-negative R_1, R_2 in $C^2(\mathbb{T})$ such that

$$R(k,k') = R_1(k)R_2(k'), \quad R_j(k) \approx |k|^{\beta_j}, \quad j = 1,2;$$

[I]: there exists $\beta_3 > 0$ such that

$$S(k) := \frac{|\omega'(k)|}{\gamma R_1(k)} \approx \frac{\cos(\pi k)}{|k|^{\beta_3}} \quad k \neq 0;$$

[ND]: the transmission at the interface does not degenerate on $\mathbb{T}$:

$$\inf_{k\in\mathbb{T}} p_+(k) \neq 0 \quad \text{and} \quad p_+^* := \lim_{k\to 0} p_+(k) > 0;$$

[D]: the probability of absorption $p_0(k)$ has a logarithmic decay at 0:

$$\lim_{k\to 0^+} |\log k|^{\kappa} p_0(k) > 0;$$

[P]: We have that $\beta_1 \leq 1 + \beta_2$ and $\beta_3 < 1 + \beta_2 < 2\beta_3$:

$$\alpha := \frac{1+\beta_2}{\beta_3} \in (1,2).$$

Associated function spaces

In order to state our main result, we need to introduce the following functional space:

- the space $\mathcal{C}_{T_o}$ is composed by all the functions ϕ in $C_b(\mathbb{R}_* \times \mathbb{T}_*)$ satisfying (IC) and that can be continuously extended to $\bar{\mathbb{R}}_\iota \times \mathbb{T}_*$, for $\iota \in \{+, -\}$;
- the space $\mathcal{H}_o$ is the completion of $C_c^\infty(\mathbb{R}_*)$ under the seminorm $\|\cdot\|_{\mathcal{H}_o}$ given by

$$\|u\|_{\mathcal{H}_o} := \hat{\mathcal{E}}^{1/2}[u],$$

for any Borel function $u\colon \mathbb{R}_* \to \mathbb{R}$ such that the expression is finite, with the following Dirichlet form:

$$\hat{\mathcal{E}}[u] := \frac{1}{2}\int_{\mathbb{R}^2} (u(y') - u(y))^2 q_\alpha(y'-y)\left(\mathbb{1}_{yy'>0} + p_+^*\mathbb{1}_{yy'<0}\right) dydy' + \frac{1}{2}\int_{\mathbb{R}^2} (u(-y') - u(y))^2 q_\alpha(y'-y) p_-^* \mathbb{1}_{yy'<0}\, dydy'.$$

Main result

Let W_0 be in $\mathcal{C}_{T_o}$ such that $\bar{W}_0 - T_o \in \mathcal{H}_o$ and $\bar{W}_0 - T_\infty \in L^2(\mathbb{R})$ for some

$$T_\infty \in \mathbb{R}, \quad \text{where} \quad \bar{W}_0(y) := \int_{\mathbb{T}} W_0(y,k) \frac{R_2(k)}{R_1(k)}\, dk.$$

Let $W_\lambda(t,y,k)$ be the classical solution to rescaled interface problem. Then,

$$\lim_{\lambda \to +\infty} \langle W_\lambda(t), F \rangle_{L^2(\mathbb{R}\times\mathbb{T})} = \langle \bar{W}(t), F \rangle_{L^2(\mathbb{R}\times\mathbb{T})}, \quad F \in C_c^\infty(\mathbb{R}\times\mathbb{T}).$$

Moreover, the limit function $\bar{W}$ is the weak solution to

(LE):

$$\partial_t \bar{W}(t,y) = -\bar{\gamma}|\partial_{yy}^2|^{\alpha/2} \bar{W}(t,y) - \bar{\gamma} p_- \,\mathrm{p.v.} \int_{yy'>0} q_\alpha(y'-y)[\bar{W}(t,y') - \bar{W}(t,-y')]\, dy'$$

with boundary conditions

$$\bar{W}(0,y) = \bar{W}_0(y); \qquad \bar{W}(t,0) = T_o.$$

Above, the fractional diffusion coefficient is given by

$$\bar{\gamma} := \frac{\gamma R_2^* S_*^{1+\alpha}}{S_*'} \left(\int_{\mathbb{T}} \frac{R_2(k)}{R_1(k)}\, dk \right)^{-1} \int_0^{+\infty} \tau^\alpha e^{-\tau}\, d\tau.$$

Sketch of the proof I

Probabilistic interpretation

- (rKE) is the Kolmogorov equation of processes $\{K_\lambda(t,k), Y_\lambda(t,y,k)\}_{t\geq 0}$;
- Add interface conditions to processes $\rightsquigarrow \{K^o_\lambda(t,k), Y^o_\lambda(t,y,k)\}_{t\geq 0}$;
- Similarly, (LE) is the Kolmogorov equation of a jump process $\{\eta^o(t,y)\}_{t\geq 0}$;

As λ goes to $+\infty$, the processes $\{Y^o_\lambda(t,y,k)\}_\lambda$ converge in finite distributions and M_1-weakly to $\eta^o(t,y)$.

Weak convergence of solutions

The solution W_λ of (rKE)+(IC) L^2-converges to the weak solution $\bar{W}$ to (LE), exploiting that:

$$W_\lambda(t,y,k) = \mathbb{E}\left[W_0\left(Y^o_\lambda(t,y,k), K^o_\lambda(t,k)\right)\right];$$
$$\bar{W}(t,y) = \mathbb{E}\left[\bar{W}_0\left(\eta^o(t,y)\right)\right].$$

- the function $t \to \|W_\lambda(t)\|_{L^2_\pi(\mathbb{R}\times\mathbb{T})}$ is non-increasing;
- for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\limsup_{\lambda\to+\infty} \|W_\lambda(\delta) - \bar{W}_0\|_{L^2_\pi(\mathbb{R}\times\mathbb{T})} < \varepsilon.$$

Sketch of the proof II: Construction of the processes

- $K_0(k) = k$ and $\{K_n(k)\}_{n\in\mathbb{N}}$ i.i.d. r.v. on $\mathbb{T}$ such that $K_1(k) \sim R_2(k)dk$;
- the process $\tilde{\mathfrak{T}}(t,k)$ as the linear interpolation between the values of

$$\mathfrak{T}_n(k) := \sum_{j=0}^{n-1} (\gamma R_1(K_j(k)))^{-1}\tau_j, \qquad \{\tau_n\}_{n\in\mathbb{N}_0} \text{ i.i.d. such that } \tau_0 \sim \exp(1);$$

- the process $\tilde{Z}(t,y,k)$ as the linear interpolation of $Z_{N(t)}(y,k)$, where

$$Z_n(y,k) := y - \sum_{j=0}^{n-1} \omega'(K_j(k))(\gamma R_1(K_j(k)))^{-1}\tau_j, \qquad N(t) \sim \mathrm{Pois}_t(1)$$

- Letting $\mathcal{S}(t,k) := \tilde{N}^{-1}(\tilde{\mathfrak{T}}^{-1}(t,k))$, we have $K(t,k) = K_{[\tilde{\mathfrak{T}}^{-1}(t,k)]}(k)$ and

$$Y(t,y,k) := \tilde{Z}(\mathcal{S}(t,k),y,k) = y - \int_0^t \omega'(K(s,k))\,ds.$$

Sketch of the proof II: Construction of the processes

- $K_0(k) = k$ and $\{K_n(k)\}_{n\in\mathbb{N}}$ i.i.d. r.v. on $\mathbb{T}$ such that $K_1(k) \sim R_2(k)dk$;
- the process $\tilde{\mathfrak{T}}_\lambda(t,k)$ as the linear interpolation between the values of

$$\mathfrak{T}_n^\lambda(k) := \frac{1}{\lambda}\sum_{j=0}^{n-1}(\gamma R_1(K_j(k)))^{-1}\tau_j, \qquad \{\tau_n\}_{n\in\mathbb{N}_0} \text{ i.i.d. such that } \tau_0 \sim \exp(1);$$

- the process $\tilde{Z}_\lambda(t,y,k)$ as the linear interpolation of $Z^\lambda_{N(\lambda t)}(y,k)$, where

$$Z_n^\lambda(y,k) := y - \frac{1}{\lambda^{1/\alpha}}\sum_{j=0}^{n-1}\omega'(K_j(k))(\gamma R_1(K_j(k)))^{-1}\tau_j, \qquad N(t) \sim \mathrm{Pois}_t(1)$$

- Letting $\mathcal{S}_\lambda(t,k) := \tilde{N}_\lambda^{-1}(\tilde{\mathfrak{T}}_\lambda^{-1}(t,k))$, we have $K_\lambda(t,k) = K_{[\tilde{\mathfrak{T}}_\lambda^{-1}(t,k)]}(k)$ and

$$Y_\lambda(t,y,k) := \tilde{Z}_\lambda(\mathcal{S}_\lambda(t,k),y,k) = y - \frac{1}{\lambda^{1/\alpha-1}}\int_0^t \omega'(K_\lambda(s,k))\,ds.$$

Sketch of the proof II: Construction of the processes

- a sequence of stopping times as $n_0 := 0$ and then

$$n_{m+1} := \inf\{n > n_m : (-1)^m Z_n(y,k) < 0\}, \quad m \in \mathbb{N}.$$

- a sequence $\{\sigma_m\}_{m\in\mathbb{N}}$ of $\{\pm 1, 0\}$-valued r.v. that are indipendent when conditioned on $\{K_n(k)\}_{n\geq 0}$ and

$$\mathbb{P}\left(\sigma_m = \iota | \{K_n(k)\}_{n\geq 0}\right) = p_\iota(K_{n_m - 1}(k)), \quad \iota \in \{0, \pm 1\}.$$

- Then,

$$Z_n^o(y,k) := \left(\prod_{j=1}^m \sigma_j\right) Z_n(y,k), \qquad \text{for } n_m \leq n \leq n_{m+1}.$$

- A similar construction holds for processes $Y^o(t,y,k)$, $\tilde{Z}^o(t,y,k)$ and $K^o(s,k)$.
- Introducing scaling $\lambda > 0$ and $\theta := \mathbb{E}[(\gamma R_1(K_1(k)))^{-1}]$,

$$\lim_{\lambda\to+\infty} \sup_{t\in[0,t_*]} |\mathcal{S}(t) - t/\theta| = 0, \quad \mathbb{P}\text{-a.s.}$$

Sketch of the proof II: Construction of the processes

- a sequence of stopping times as $n_0^\lambda := 0$ and then

$$n_{m+1}^\lambda := \inf\left\{n > n_m^\lambda \colon (-1)^m Z_n^\lambda(y,k) < 0\right\}, \quad m \in \mathbb{N}.$$

- a sequence $\{\sigma_m^\lambda\}_{m\in\mathbb{N}}$ of $\{\pm 1, 0\}$-valued r.v. that are indipendent when conditioned on $\{K_n(k)\}_{n\geq 0}$ and

$$\mathbb{P}\left(\sigma_m^\lambda = \iota | \{K_n(k)\}_{n\geq 0}\right) = p_\iota(K_{n_m^\lambda - 1}(k)), \quad \iota \in \{0, \pm 1\}.$$

- Then,

$$Z_n^{o,\lambda}(y,k) := \left(\prod_{j=1}^m \sigma_j^\lambda\right) Z_n^\lambda(y,k), \qquad \text{for } n_m^\lambda \leq n \leq n_{m+1}^\lambda.$$

- A similar construction holds for processes $Y_\lambda^o(t,y,k)$, $\tilde{Z}_\lambda^o(t,y,k)$ and $K_\lambda^o(s,k)$.
- Introducing scaling $\lambda > 0$ and $\theta := \mathbb{E}[(\gamma R_1(K_1(k)))^{-1}]$,

$$\lim_{\lambda\to+\infty} \sup_{t\in[0,t_*]} |\mathcal{S}_\lambda(t) - t/\theta| = 0, \quad \mathbb{P}\text{-a.s.}$$

Sketch of the proof III: Semigroup analysis

The weak convergence of $\tilde{Z}^o_\lambda(t,y,k)$ to $\zeta^o(t,y) := \eta^o(\theta t, y)$ follows from:

Let us consider the corresponding Markov semigroups:

$$P^{o,\lambda}_t u(y) := \mathbb{E}[u(\tilde{Z}^o_\lambda(t,y,k)),\ t < \mathfrak{s}^\lambda_{y,\mathfrak{f}}];$$
$$P^o_t u(y) := \mathbb{E}[u(\zeta^o(t,y)),\ t < \mathfrak{t}_{y,\mathfrak{f}}].$$

As λ tends to $+\infty$, the semigroups $\{P^{o,\lambda}_t\}_{\lambda>0}$ are strongly L^2-convergent, uniformly on compact time intervals, to P^o_t.

⚠ The process $Z^o_\lambda(t,y,k)$ it is actually not Markovian but it is possible to construct a Markov process $\{\hat{Z}^o(t,y)\}_{t\geq 0}$ that is "close" to it, in the sense:

$$\mathbb{P}\left(yZ^\lambda_1(y,k) < 0\right) \leq \exp\left\{-\frac{|y|\lambda^{1/\alpha}}{|S(k)|}\right\},$$

where $Z^\lambda_1(y,k)$ is the particle position at the time of the first jump, starting in y.

Sketch of the proof IV: Dirichlet forms

To show the the L^2-convergence of the semigroups, we rely on the corresponding Dirichlet forms:

$$\mathcal{E}^o[u] := \lim_{t\to 0+} \frac{1}{t} \int_{\mathbb{R}} [u(y) - P^o_t u(y)]\, u(y)\, dy \approx \hat{\mathcal{E}}[u].$$

Mosco convergence for Dirichlet forms

A family of Dirichlet forms $\mathcal{E}_\lambda$ is M-convergent to a Dirichlet form $\mathcal{E}_\infty$, as $\lambda \to +\infty$, if for any $u \in L^2(\mathbb{R})$:

- for any $\{u_\lambda\}_{\lambda>0}$ weakly convergent to u in $L^2(\mathbb{R})$, it holds that

$$\liminf_{\lambda\to+\infty} \mathcal{E}_\lambda[u_\lambda] \geq \mathcal{E}_\infty[u]$$

- there exists $\{v_\lambda\}_{\lambda>0}$ strong convergent to u in $L^2(\mathbb{R})$ such that

$$\limsup_{\lambda\to+\infty} \mathcal{E}_\lambda[v_\lambda] \leq \mathcal{E}_\infty[u].$$

[M94]: The Dirichlet forms $\mathcal{E}_\lambda$ M-converge to $\mathcal{E}_\infty$ if and only if the associated Markov semigroups strongly L^2-converges, uniformly on compact intervals.

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Thank you for your attention!