# (Weighted) Parabolic operators: fractional powers and the Kato square root problem

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# Parabolic operators with complex coefficients

$$\mathcal{H} = \partial_t - (w(x))^{-1} \operatorname{div}_x(A(x,t)\nabla_x), \ (x,t) \in \mathbb{R}^{n+1},$$

$A = A(x,t) = \{A_{i,j}(x,t)\}_{i,j=1}^n,$

$$c_1|\xi|^2 w(x) \leq \operatorname{Re}(A(x,t)\xi \cdot \bar{\xi}), \ |A(x,t)\xi \cdot \zeta| \leq c_2 w(x)|\xi||\zeta|,$$

for some $c_1, c_2 \in (0,\infty)$ and for all $\xi, \zeta \in \mathbb{C}^n$, $(x,t) \in \mathbb{R}^{n+1}$.

The weight $w = w(x)$ is a real-valued function belonging to the Muckenhoupt class $A_2(\mathbb{R}^n, dx)$,

$$\left(\fint_Q w \, dx\right)\left(\fint_Q w^{-1} \, dx\right) \leq [w]_{A_2},$$

for all cubes $Q \subset \mathbb{R}^n$.

# Outline and summary

Fractional powers of (unweighted) operators ($s \in (0,1)$)

$$\mathcal{H}^s = (\partial_t - \mathrm{div}_x(A(x,t)\nabla_x))^s,\ (x,t) \in \mathbb{R}^n \times \mathbb{R}.$$

$$\begin{aligned}(\lambda^{1-2s}\mathrm{u}')'(\lambda) &= \lambda^{1-2s}\mathcal{H}\mathrm{u}(\lambda), \quad \lambda \in (0,\infty),\\ \mathrm{u}(0) &= u,\\ -\lim_{\lambda\downarrow 0} \lambda^{1-2s}\mathrm{u}'(\lambda) &= c_s\mathcal{H}^s u.\end{aligned}$$

The Kato square root problem for weighted operators

$$\|\sqrt{\mathcal{H}}\, u\|_{2,\mu} \sim \|\nabla_x u\|_{2,\mu} + \|D_t^{1/2} u\|_{2,\mu} \qquad (u \in \mathrm{E}_\mu(\mathbb{R}^{n+1})).$$

$$\partial_t - x\cdot\nabla_y - (w(x))^{-1}\,\mathrm{div}_x(A(x,y,t)\nabla_x),\ (x,y,t) \in \mathbb{R}^{2m+1}.$$

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# The fractional Laplacian $(-\Delta_x)^s$ in $\mathbb{R}^n$

Potential theory, singular integrals and harmonic analysis, the theory of Lévy/stable processes, degenerate elliptic equations, .....

If $s \in (0,1)$, and if $\mathcal{U}$ solves

$$\partial_\lambda(\lambda^{1-2s}\partial_\lambda\mathcal{U})(\lambda,x) = -\lambda^{1-2s}\Delta_x\mathcal{U}(\lambda,x),$$
$$\mathcal{U}(0,x) = u(x),$$

where $u \in \mathrm{D}((-\Delta_x)^s)$, then

$$-\lim_{\lambda\to 0^+}\lambda^{1-2s}\partial_\lambda\mathcal{U}(\lambda,x) = c_s(-\Delta_x)^s u(x), \qquad x \in \mathbb{R}^n.$$

$(-\Delta_x)^s u = 0$ in a domain $\Omega \subset \mathbb{R}^n$ can be studied through linear degenerate elliptic equations.

# Operator theoretical context

Assume $A$ sectorial operator on a Banach space $X$: can one define a function space valued ODE and a solution $\mathrm{u}$,

$$(\lambda^{1-2s}\mathrm{u}'(\lambda))' = \lambda^{1-2s}A\mathrm{u}(\lambda), \quad \lambda \in (0,\infty),\ \mathrm{u}(0) = u,$$

where $u \in \mathrm{D}(A^s)$, such that

$$-\lim_{\lambda\to 0+} \lambda^{1-2s}\mathrm{u}'(\lambda) = c_s A^s u?$$

This is a linear ODE in the Banach space $X$ with initial data $u \in X$ which degenerates for $\lambda = 0$, unless $s = 1/2$.

Problems: existence and uniqueness, properties of the D2N map, the relation between the D2N map and $c_s A^s$.

# Parabolic versions of $(-\Delta_x)^s$ in $\mathbb{R}^{n+1}$

$\partial_t + (-\Delta_x)^s$: Lévy/stable processes, option pricing models, ....

$\partial_t^\beta + (-\Delta_x)$ and $\partial_t^\beta + (-\Delta_x)^s$: stochastic processes with memory, ....

$(\partial_t - \Delta_x)^s$: closely connected to Continuous Time Random Walks (CTRWs) - random jumps in space are coupled with random waiting times, anomalous diffusions, ....

The stochastic processes related to $(\partial_t - \Delta_x)^s$ differ from those related to $\partial_t + (-\Delta_x)^s$ and $\partial_t^\beta + (-\Delta_x)^s$: for the latter jumps in space are independent of (the waiting) time(s).

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# The fractional heat operator

$$\widehat{\mathcal{H}^s u}(\xi,\tau) := (i\tau + |\xi|^2)^s \widehat{u}(\xi,\tau).$$

$$\mathcal{H}^s u(x,t) = \iint_{\mathbb{R}^{n+1}} (u(x,t) - u(x+w,t+\tau))K_{n,s}(w,-\tau)\,dwd\tau,$$

with kernel

$$K_{n,s}(w,\tau) := \frac{1}{\Gamma(-s)}\frac{W_n(w,\tau,0,0)}{\tau^{1+s}}.$$

$$\lim_{s\nearrow 1} \mathcal{H}^s u(x,t) = (\partial_t - \Delta_x)u(x,t).$$

# An extension problem related to $(\partial_t - \Delta_x)^s$

Together with O. Sande we discovered the parabolic analogue of the result of Caffarelli and Silvestre.

If $s \in (0,1)$, and if $\mathcal{U}$ now solves

$$\begin{aligned} \partial_\lambda(\lambda^{1-2s}\partial_\lambda \mathcal{U})(\lambda, x, t) &= \lambda^{1-2s}(\partial_t - \Delta_x)\mathcal{U}(\lambda, x, t), \\ \mathcal{U}(0, x, t) &= u(x, t), \; (\lambda, x, t) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}, \end{aligned}$$

where $u \in \mathrm{D}((\partial_t - \Delta_x)^s)$, then

$$-\lim_{\lambda \to 0^+} \lambda^{1-2s}\partial_\lambda \mathcal{U}(\lambda, x, t) = c_s(\partial_t - \Delta_x)^s u(x, t), \; (x, t) \in \mathbb{R}^n \times \mathbb{R}.$$

$(\partial_t - \Delta_x)^s u = 0$ in a domain $\Omega \times J \subset \mathbb{R}^n \times \mathbb{R}$ can be studied through linear degenerate parabolic equations.

# The fractional heat operator and the extension

$$P_\tau u(x,t) := \int_{\mathbb{R}^n} W_n(x,\tau,w,0)u(w,t-\tau)\,dw,\ \ \tau>0.$$

$$\mathcal{H}^s u(x,t) = -\frac{s}{\Gamma(1-s)}\int_0^\infty \tau^{-1-s}\big(P_\tau u(x,t)-u(x,t)\big)\,d\tau.$$

$$U(\lambda,x,t) := \frac{1}{2^{2s}\Gamma(s)}\lambda^{2s}\int_0^\infty \frac{1}{\tau^{1+s}}e^{-\frac{\lambda^2}{4\tau}}P_\tau u(x,t)\,d\tau,$$

$$-\lim_{\lambda\to 0^+}\lambda^{1-2s}\partial_\lambda U(\lambda,x,t) = c_s\mathcal{H}^s u(x,t), \qquad (x,t)\in\mathbb{R}^n\times\mathbb{R}.$$

Generalizations to $\partial_t+\mathcal{L}_x$, $\mathcal{L}_x = -\operatorname{div}_x(A(x)\nabla_x)$, with $A$ real, bounded, uniformly elliptic and symmetric.

# Fractional powers of parabolic operators with time-dependent measurable coefficients

With M. Litsgård we consider fractional operators of the form

$$\mathcal{H}^s = (\partial_t - \mathrm{div}_x(A(x,t)\nabla_x))^s, \ (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

where $s \in (0,1)$ and $A = A(x,t) = \{A_{i,j}(x,t)\}_{i,j=1}^n$ is an accretive, bounded, complex, measurable, $n \times n$-dimensional matrix valued function.

Assuming that $A = A(x,t) = \{A_{i,j}(x,t)\}_{i,j=1}^n$ is real, we derive some local properties of solutions to

$$\mathcal{H}^s u = (\partial_t - \mathrm{div}_x(A(x,t)\nabla_x))^s u = 0 \text{ for } (x,t) \in \Omega \times J,$$
$$u = f \text{ for } (x,t) \in \mathbb{R}^{n+1} \setminus (\Omega \times J).$$

We allow for non-symmetric and time-dependent coefficients.

# Definition of $\mathcal{H}$

Let $H := \mathrm{L}^2(\mathbb{R}^{n+1}) := \mathrm{L}^2(\mathbb{R}^{n+1}, dxdt)$. We introduce the inhomogeneous energy space $\mathrm{E}(\mathbb{R}^{n+1})$ equipped with the Hilbertian norm

$$\|v\|_{\mathrm{E}(\mathbb{R}^{n+1})} := \left(||v||_2^2 + ||\nabla_x v||_2^2 + ||D_t^{1/2} v||_2^2\right)^{1/2}.$$

For short we have the triple

$$H = \mathrm{L}^2(\mathbb{R}^{n+1}),\ V := \mathrm{E}(\mathbb{R}^{n+1}),\ V' := \mathrm{E}(\mathbb{R}^{n+1})^*,$$

where $V' = \mathrm{E}(\mathbb{R}^{n+1})^*$ is the (anti)-dual of $V = \mathrm{E}(\mathbb{R}^{n+1})$. Then

$$V \hookrightarrow H \hookrightarrow V'.$$

# Definition of $\mathcal{H}$

$$\partial_t = D_t^{1/2} H_t D_t^{1/2} (= |\tau|^{1/2} i\mathrm{sign}(\tau)|\tau|^{1/2}).$$

$$c_1|\xi|^2 \leq \mathrm{Re}(A(x,t)\xi \cdot \bar{\xi}), \ |A(x,t)\xi \cdot \zeta| \leq c_2|\xi||\zeta|.$$

$$\mathcal{E}(u,v) := \iint_{\mathbb{R}^{n+1}} A(x,t)\nabla_x u \cdot \overline{\nabla_x v} + H_t D_t^{1/2} u \cdot \overline{D_t^{1/2} v}\, dxdt.$$

The form induces a bounded operator $\mathcal{H}$ from $V$ into $V'$ via

$$\langle \mathcal{H}u, v\rangle_{V',V} := \mathcal{E}(u,v), \ u,v \in V.$$

$\mathcal{H}$ is initially an unbounded operator on $H$:

$$\mathrm{D} := \{u \in V : \mathcal{H}u \in H\}.$$

# Discovering hidden coercivity

Consider the modified sesquilinear form

$$
\begin{aligned}
a_\delta(u,v) &= \iint_{\mathbb{R}^{n+1}} A(x,t)\nabla_x u \cdot \overline{\nabla_x (1+\delta H_t) v}\, dxdt \\
&\quad + \iint_{\mathbb{R}^{n+1}} H_t D_t^{1/2} u \cdot \overline{D_t^{1/2}(1+\delta H_t) v}\, dxdt,
\end{aligned}
$$

where $\delta > 0$ is a (real) degree of freedom.

If we fix $\delta > 0$ small enough, then

$$
\operatorname{Re} a_\delta(u,u) \geq (c_1 - c_2\delta)\|\nabla_x u\|_2^2 + \delta \|H_t D_t^{1/2} u\|_2^2
$$

where $c_1, c_2$ are the ellipticity constants for $A$.

# Maximal accretivity and sectoriality: definition of $\mathcal{H}^s$

**Theorem**

*$(\mathcal{H}, \mathrm{D})$ is maximal accretive: $\mathcal{H}$ is closed and for every $\sigma \in \mathbb{C}$ with $\mathrm{Re}\,\sigma < 0$, the operator $\sigma - \mathcal{H}$ is invertible on $H$ and $(\sigma - \mathcal{H})^{-1}$ satisfies the estimate $\|(\sigma - \mathcal{H})^{-1}\|_{H \to H} \leq (|\,\mathrm{Re}\,\sigma|)^{-1}$.*

$\mathcal{H}$ has a bounded $H^\infty$ calculus and the powers $\mathcal{H}^s$ can be defined abstractly by functional calculus.

The fractional powers $\mathcal{H}^s$, $s \in (0,1)$, have the Balakrishnan representation

$$\mathcal{H}^s u := \frac{\sin(s\pi)}{\pi} \int_0^\infty \lambda^{s-1}(\lambda + \mathcal{H})^{-1}\mathcal{H}u \, d\lambda,$$

for $u \in \mathrm{D}$.

# The Kato square root estimate - Auscher-Egert-N

## Theorem

*Given* $\mathcal{H} = \partial_t - \operatorname{div}_x A(x,t)\nabla_x$ *arises from an accretive form,* $\mathrm{D}(\sqrt{\mathcal{H}}) = V$, *and*

$$\|\sqrt{\mathcal{H}}\, u\|_2 \sim \|\nabla_x u\|_2 + \|D_t^{1/2} u\|_2 \qquad (u \in V).$$

The Kato estimate - key square/quadratic function estimates:

$$(i) \int_0^\infty \iint_{\mathbb{R}^{n+1}} |\lambda(I + \lambda^2\mathcal{H})^{-1}\mathcal{H} f|^2 \,\frac{dxdtd\lambda}{\lambda} \sim ||f||_{\dot{\mathrm{E}}} \quad (f \in \mathrm{E}),$$

$$(ii) \int_0^\infty \|\lambda PM(1 + \lambda^2 PMPM)^{-1} h\|_2^2 \,\frac{d\lambda}{\lambda} \sim \|h\|_2^2 \quad (h \in \overline{\mathrm{R}(P)}).$$

The Kato square root estimate: the case $A^* = A$ does not follow from abstract functional analysis as $\mathcal{H}$ not self-adjoint.

# The domain of $\mathcal{H}^s$

The domain of $\mathcal{H}^s$, $\mathrm{D}(\mathcal{H}^s)$, is the space $\{u \in H : \ \mathcal{H}^s u \in H\}$,

$$\|u\|_{\mathrm{D}(\mathcal{H}^s)} := \|u\|_2 + \|\mathcal{H}^s u\|_2.$$

If $0 < s_1 \leq s_2 < 1$ then $\mathrm{D} \subset \mathrm{D}(\mathcal{H}^{s_2}) \subseteq \mathrm{D}(\mathcal{H}^{s_1}) \subset H$ and D is a core for $\mathrm{D}(\mathcal{H}^s)$ for all $s \in (0,1)$.

$$\mathrm{D}(\mathcal{H}^{1/2}) = V \text{ and } \|u\|_{\mathrm{D}(\mathcal{H}^{1/2})} \sim \|u\|_2 + \|\nabla_x u\|_2 + \|D_t^{1/2} u\|_2.$$

$\mathrm{H}^s_{\partial_t - \Delta_x}$: $u \in H$, $\|\mathcal{F}^{-1}((|\xi|^2 + i\tau)^{s/2}\mathcal{F}u)\|_2 < \infty$.

If $s \in (0, 1/2]$, then $\mathrm{D}(\mathcal{H}^s) = \mathrm{H}^{2s}$ and

$$\|\mathcal{H}^s u\|_2 \sim \|\mathcal{F}^{-1}((|\xi|^2 + i\tau)^s \mathcal{F}u)\|_2 \qquad (u \in \mathrm{H}^{2s}).$$

Situation less clear for $s \in (1/2, 1]$.

# Semigroup theory

$-\mathcal{H}$ generates a strongly continuous semigroup of contractions, $\mathcal{S} : [0, \infty) \to \mathcal{L}(H, H)$, on $H$, and

$$-\mathcal{H}u = \lim_{\lambda \downarrow 0} \frac{(S(\lambda) - I)u}{\lambda}, \; u \in \mathrm{D}\,.$$

By complex interpolation,

$$\mathrm{D}(\mathcal{H}^s) = [H, \mathrm{D}]_s, \text{ and}$$

$$||(\mathcal{S}(\lambda) - I)u||_2 \leq c\lambda^s(||u||_2 + ||\mathcal{H}^s u||_2) < \infty, \; \forall u \in \mathrm{D}(\mathcal{H}^s).$$

$$\mathcal{H}^s u = (\Gamma(-s))^{-1} \int_0^\infty \lambda^{-s-1}(\mathcal{S}(\lambda) - I)u \, d\lambda, \; u \in \mathrm{D}\,.$$

# The extension problem via semigroup theory

We say that $\mathrm{u}(\lambda)$ is a solution to

$$(\lambda^{1-2s}\mathrm{u}')'(\lambda) = \lambda^{1-2s}\mathcal{H}\mathrm{u}(\lambda), \quad \lambda \in (0,\infty),\ \mathrm{u}(0) = u, \tag{0.1}$$

if,

$$\mathrm{u}(\cdot) \in C^0\big([0,\infty), \mathrm{D}\big) \cap C^\infty\big((0,\infty), \mathrm{D}\big),$$
$$\langle(\lambda^{1-2s}\mathrm{u}')'(\lambda), v\rangle_H = \lambda^{1-2s}\langle\mathcal{H}\mathrm{u}(\lambda), v\rangle_{V',V},$$

for all $v \in V$, $\lambda \in (0,\infty)$, and

$$\lim_{\lambda\downarrow 0} \mathrm{u}(\lambda) = u \text{ in } H.$$

# The extension problem via semigroup theory

## Theorem

*Let* $s \in (0,1)$ *and define* $\mathcal{U}\colon [0,\infty) \to \mathcal{L}(\mathrm{D}) := \mathcal{L}(\mathrm{D},\mathrm{D})$ *as*

$$\mathcal{U}(\lambda) := \frac{1}{\Gamma(s)}\left(\frac{\lambda}{2}\right)^{2s}\int_0^\infty r^{-s}e^{-\frac{\lambda^2}{4r}}\mathcal{S}(r)\,\frac{dr}{r}.$$

*Let* $\mathrm{u}(\lambda) := \mathcal{U}(\lambda)u$*,* $u \in \mathrm{D}$*. Then* $\mathrm{u}(\lambda)$ *is a solution to* (0.1)*,*

$$\lim_{\lambda\to\infty}\langle \mathrm{u}(\lambda), v\rangle_H = 0, \text{ for all } v \in H,$$

$$||\lambda^{1-2s}\mathrm{u}'(\lambda)||_2 \leq c_\varepsilon \max\{1, |\lambda|^{2\varepsilon}\}(||u||_2 + ||\mathcal{H}^{s+\varepsilon}u||_2)) < \infty,$$

*whenever* $\lambda \in (0,\infty)$*, if* $\varepsilon \in (0,1)$ *is small,* $s+\varepsilon \leq 1$*, and*

$$-\lim_{\lambda\downarrow 0}\lambda^{1-2s}\mathrm{u}'(\lambda) = -\lim_{\lambda\downarrow 0}\lambda^{1-2s}\frac{(\mathrm{u}(\lambda)-\mathrm{u}(0))}{\lambda} = c_s\mathcal{H}^s u \text{ in } H.$$

# Connections to reinforced weak solutions

We say that $u \in \mathrm{D}(\mathcal{H}^s)$ is a solution to $\mathcal{H}^s u = 0$ in $\Omega \times J$ if $\langle \mathcal{H}^s u, \phi \rangle_H = 0$ for all $\phi \in C_0^\infty(\Omega \times J)$.

$$\mathcal{U}(\lambda, x, t) := \mathcal{U}(\lambda) u(x,t), \ (\lambda, x, t) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}, \ u \in \mathrm{D}\,.$$

We introduce $\tilde{\mathcal{U}}$ on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ through

$$\tilde{\mathcal{U}}(\lambda, x, t) := \mathcal{U}(\lambda, x, t) \text{ for } \lambda \geq 0,$$
$$\tilde{\mathcal{U}}(\lambda, x, t) := \mathcal{U}(-\lambda, x, t) \text{ for } \lambda < 0.$$

Then $\tilde{\mathcal{U}}$ is a (weighted) reinforced weak solution to the PDE

$$\partial_\lambda(|\lambda|^{1-2s}\partial_\lambda \tilde{\mathcal{U}}) = |\lambda|^{1-2s}(\partial_t \tilde{\mathcal{U}} - \mathrm{div}_x(A(x,t)\nabla_x \tilde{\mathcal{U}})).$$

# Local regularity in the case of real coefficients

Assume that $A = A(x,t) = \{A_{i,j}(x,t)\}_{i,j=1}^n$ is real, measurable. Let $(z_0,\tau_0) \in \mathbb{R}^{n+1}$. $s \in (0,1)$, $\varepsilon \in (0,1)$ small, $s+\varepsilon \leq 1$, $\bar{s} := \max\{1/2, s+\varepsilon\}$. Assume that $u \in \mathrm{D}(\mathcal{H}^{\bar{s}})$ is a solution to $\mathcal{H}^s u = 0$ in $Q_{4r}(z_0,\tau_0)$, that $\|u\|_{\mathrm{L}^\infty(\mathbb{R}^n \times (-\infty,\tau_0+16r^2])} < \infty$.

## Theorem

*Assume that* $u \geq 0$ *on* $\mathbb{R}^n \times (-\infty, \tau_0 + 16r^2]$*.*

$$\sup_{Q_{2r}^-(z_0,\tau_0)} u \leq c \inf_{Q_{2r}^+(z_0,\tau_0)} u.$$

## Theorem

*If* $(x,t), (y,s) \in Q_r(z_0,\tau_0)$*, then*

$$|u(x,t) - u(y,s)| \leq \big(d(x,t,y,s)/r\big)^\alpha \|u\|_{\mathrm{L}^\infty(\mathbb{R}^n \times (-\infty,\tau_0+16r^2])}.$$

# Local regularity in the case of real coefficients

The $C_0$-semigroup $\mathcal{S}(\lambda)$ generated by $-\mathcal{H}$ can be identified, following the proof of Hille, as

$$\mathcal{S}(\lambda)u = \lim_{m\to\infty}\left(I + \frac{\lambda}{m}\mathcal{H}\right)^{-m} u$$

for $\lambda > 0$ and for all $u \in H$.

**Lemma**

*$(1+\sigma^{-1}\mathcal{H})^{-m}$, is represented by an non-negative integral kernel $K_{\sigma,m}$ with $|K_{\sigma,m}(x,t,y,s)|$ bounded by*

$$C\sigma^m \chi_{(0,\infty)}(t-s)(t-s)^{-n/2+m-1} e^{-\sigma(t-s)} e^{-c\frac{|x-y|^2}{t-s}}.$$

# Remarks

Considering $\mathcal{H}^s$ we would like to only assume $u \in \mathrm{D}(\mathcal{H}^s)$ to make conclusions.

In the case $s = 1/2$, the results presented can be sharpened. If $u \in \mathrm{D}(\mathcal{H}^{1/2})$ then $\mathrm{u}(\lambda) := e^{-\lambda\sqrt{\mathcal{H}}}u$ is a reinforced weak solution to the PDE, and all results hold for $u \in \mathrm{D}(\mathcal{H}^{1/2})$.

In the case $s \in (0, 1/2)$, we demand $u \in \mathrm{D}(\mathcal{H}^{1/2})$, which is considerably more a priori regularity on $u$: we construct reinforced weak solutions.

In the case $s \in (1/2, 1)$, we assume $u \in \mathrm{D}(\mathcal{H}^{s+\varepsilon})$, where $\varepsilon > 0$ can be chosen arbitrary small, but fixed. We need this to have $||\lambda^{1-2s}\mathrm{u}'(\lambda)||_2 < \infty$.

# The Kato square root problem for weighted operators

Together with A. Ataei and M. Egert we solve the Kato square root problem for the weighted parabolic operator $\mathcal{H}$.

**Theorem**

*The part of $\mathcal{H}$ in $\mathrm{L}^2_\mu(\mathbb{R}^{n+1})$, with maximal domain*

$$\mathrm{D}(\mathcal{H}) = \{u \in \mathrm{E}_\mu(\mathbb{R}^{n+1}) : \mathcal{H}u \in \mathrm{L}^2_\mu(\mathbb{R}^{n+1})\},$$

*is maximal accretive. The square root of $\mathcal{H}$ is well-defined, the domain of the square root is that of the accretive form, that is, $\mathrm{D}(\sqrt{\mathcal{H}}) = \mathrm{E}_\mu(\mathbb{R}^{n+1})$, and*

$$\|\sqrt{\mathcal{H}}\, u\|_{2,\mu} \sim \|\nabla_x u\|_{2,\mu} + \|D_t^{1/2} u\|_{2,\mu} \qquad (u \in \mathrm{E}_\mu(\mathbb{R}^{n+1})).$$

*The same conclusions are true with $\mathcal{H}$ replaced by $\mathcal{H}^*$.*

Our proof is novel in several respects compared to [AEN].

- The proof is based purely on second order techniques in contrast to [AEN].
- By only treating a few terms differently, the argument in my original paper [KN] proves the parabolic Kato square problem in full generality.

- We do not need any off-diagonal estimates for $D_t^{1/2}u$.
- We can essentially use the same test functions in the local *Tb* theorem as in the original solution of (elliptic) Kato problem.

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