A Yosida's parametrix approach to Varadhan's estimates for a *weakly* hypoelliptic diffusion

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(joint work with Sergio Polidoro)

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Kolmogorov Operators and their Applications

General framework

Non-divergence form parabolic operator:

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(t,x)\partial_{x_i x_j} + \sum_{i=1}^{n} \mu_i(t,x)\partial_{x_i} + \partial_t, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^n,$$

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Backward Kolmogorov operator of:

$$\mathrm{d}X_s = \mu(t,X_s)\mathrm{d}s + \sigma(t,X_s)\mathrm{d}W_s, \qquad X_t = x,$$

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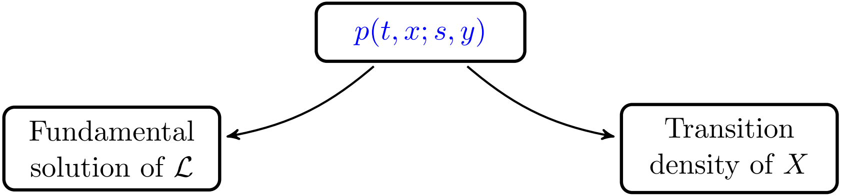

Parametrix method: general construction - I

Levi (1907)

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- $H = H(t, x; T, y)$ *parametrix function*, such that

$$
\begin{aligned}
&- \quad H(t, x; T, \cdot) \to \delta_x, \qquad \text{as } t \to T^- \\
&- \quad H \text{ sufficiently regular}
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- seek p of the form

$$p(t, x; T, y) = H(t, x; T, y) + \underbrace{\int_t^T \int H(t, x; s, \xi)\Phi(s, \xi; T, y)\mathrm{d}\xi\mathrm{d}s}_{\text{remainder}}$$

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- formally write

$$\mathcal{L}p(\cdot,\cdot;T,y) = 0$$

Parametrix method: general construction - II

- Fixed-point equation for Φ:

$$\Phi(t,x;T,y) = \mathcal{L}H(t,x;T,y) + \int_t^T \int \mathcal{L}H(t,x;T,y)\,\Phi(s,\xi;T,y)\mathrm{d}\xi\mathrm{d}s$$

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$$\Phi = \sum_{n=0}^{\infty} K_n$$

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$$K_2(t,x;T,y) = \int_t^T \int \mathcal{L}H(t,x;T,y)\,K_1(s,\xi;T,y)\mathrm{d}\xi\mathrm{d}s, \qquad \text{etc.}$$

(Uniformly) parabolic case - I

Coercivity condition:

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j \leq \Lambda|\xi|^2, \qquad t>0, \quad x,\xi \in \mathbb{R}^n$$

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$$\Big\downarrow \text{ freeze coefficients at } (s,\xi)$$

$$\mathcal{L}^{s,\xi} = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(s,\xi)\partial_{x_i x_j} + \partial_t$$

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$$\mathcal{L}^{s,\xi} = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(s,\xi)\partial_{x_i x_j} + \partial_t$$

is a heat-type operator $\rightarrow$ Gaussian fundamental solution:

$$\Gamma_{s,\xi}(t,x;T,y) = \frac{1}{(2\pi)^{n/2}\sqrt{\det \mathbf{C}_{s,\xi}(T-t)}} e^{-\frac{1}{2}\langle \mathbf{C}_{s,\xi}^{-1}(T-t)(y-x),y-x\rangle}$$

$$\big(\mathbf{C}_{s,\xi}(\tau)\big)_{i,j} = a_{i,j}(s,\xi)\,\tau \qquad \text{covariance matrix}$$

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- **Not sharp in the logarithm**

Ultra-parabolic case

$$\frac{1}{C}\, p_{\frac{1}{C}}(t,x;T,y) \leq p(t,x;T,y) \leq C \underbrace{p_C(t,x;T,y)}_{\text{fund. sol. of } \mathcal{K}_C}$$

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Full expansions: Molchanov (1975), Azencott (1984), (WKB)

$$p(t,x;T,y) = \frac{\exp\left(-\frac{d^2(x,y)}{2(T-t)}\right)}{(T-t)^{n/2}}\Big(\alpha_0(x,y) + (T-t)\alpha_1(x,y) + \cdots\Big)$$

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Asymptotic methods in finance: Friz et al. (2015)

Sub-Riemannian case

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- Léandre (1987a), Léandre (1987b):

$$d(x, y) \longrightarrow \text{Carnot-Caratheodory metric}$$

Beyond sub-Riemannian

Key remark:

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- Ψ solution to

$$\Psi(t,x;T,y) = \min_{\omega} \int_t^T |\omega(s)|^2 \mathrm{d}s,$$

$\omega \in L^2([t,T])$ such that

$$\begin{cases} \dot\gamma(s) = \sigma\big(\gamma(s)\big)\omega(s), \quad t < s < T, \\ \gamma(t) = x, \ \gamma(T) = y \end{cases}$$

Langevin-type dynamics

$$\begin{cases} \mathrm{d}X_t^1 = \sigma\,\mathrm{d}W_t \\ \mathrm{d}X_t^2 = X_t^1\mathrm{d}t \end{cases} \longrightarrow \qquad \mathcal{K}_\sigma = \frac{\sigma^2}{2}\partial_{x_1x_1} + \underbrace{x_1\partial_{x_2} + \partial_t}_{:=Y}$$

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Explicit transition density (Komogorov 1933):

$$p(t,x;T,y) = \frac{e^{-\frac{1}{2}\Psi_\sigma(t,x;T,y)}}{2\pi\sqrt{\det \mathbf{C}}}, \quad \mathbf{C}(\sigma,s) := \sigma^2 \begin{pmatrix} s & -\frac{s^2}{2} \\ -\frac{s^2}{2} & \frac{s^3}{3} \end{pmatrix},$$

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$$\begin{aligned} \Psi_\sigma(t,x;T,y) &= \left\langle \mathbf{C}^{-1}(\sigma, T-t)\big(y - e^{B(T-t)}x\big), y - e^{B(T-t)}x \right\rangle \\ &= \frac{3\big(2(y_2-x_2) - (T-t)(x_1+y_1)\big)^2}{\sigma^2(T-t)^3} + \frac{(y_1-x_1)^2}{\sigma^2(T-t)} \end{aligned}$$

Weakly hypoelliptic case

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- assume:

$$\mathrm{Lie}(\partial_t + Z_0, Z_1, \ldots, Z_m)(x) = \mathbb{R}^{n+1} \quad \text{(weak Hörmander cond.)}$$

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$$\nexists\ d(x,y)$$

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- Ψ **survives:** now ω control such that

$$\begin{cases} \dot\gamma(s) = \mu\big(\gamma(s)\big) + \sigma\big(\gamma(s)\big)\omega(s), & t<s<T, \\ \gamma(t)=x, \; \gamma(T)=y \end{cases}$$

Asian Black-Scholes operator

$$\begin{cases} \mathrm{d}X_s^1 = \sigma X_s^1 \,\mathrm{d}W_s, & X_t^1 = x_1 > 0 \ (!) \\ \mathrm{d}X_s^2 = X_s^1 \,\mathrm{d}s, & X_t^2 = x_2 \end{cases}$$

Yor (1992), explicit density: difficult to study!

$$\mathcal{L} = \partial_t + x_1 \partial_{x_2} + \frac{\sigma^2 x_1^2}{2} \partial_{x_1 x_1}, \qquad (t, x_1, x_2) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R},$$

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- control problem: ω such that

$$\begin{cases} \dot{\gamma}_1(s) = \sigma \, \omega(s) \, \gamma_1(s) \\ \dot{\gamma}_2(s) = \gamma_1(s) \end{cases} \quad s \in]t, T[, \qquad \begin{cases} \gamma(t) = x \\ \gamma(T) = y \end{cases}$$

(!) (T, y) reachable if $t < T$ and $x_2 < y_2$

Asian options: estimates

- Cibelli et al. (2019):

Harnack chains

$$\downarrow$$

$$\frac{\kappa^{-1}}{y_1^2(T-t)^2} e^{-C\frac{\Psi(t,x;T,y)}{2}} \leq p(t,x;T,y) \leq \frac{\kappa}{y_1^2(T-t)^2} e^{-\frac{\Psi(t,x;T,y)}{2C}}$$

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- P. and Polidoro (2021):

Yosida's parametrix

$$\downarrow$$

$$C = 1$$

$$\downarrow$$

$$\log p(t,x;T,y) \sim \Psi(t,x;T,y)/2 \qquad \text{as } (T-t) \to 0^+$$

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$$\Psi(z,w) \approx \frac{4}{\sigma^2(T-t)}\left[\mathbf{h}(z,w)\left(\sqrt{\frac{y_1}{x_1}}+\sqrt{\frac{x_1}{y_1}}-2\right)+\sum_{n=2}^{N} G_n\big(\mathbf{h}(z,w)\big)\right]$$

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$$G_n(\eta) := a_n \mathbf{1}_{]1,\infty[}(\eta)\frac{(\eta-1)^n}{\eta^{n-1}} + b_n \mathbf{1}_{]0,1[}(\eta)\frac{(-\log\eta)^n}{\big(1-\log\eta\big)^{n-2}}$$

Asymptotic expansion of Ψ

- Cibelli et al. (2019): inverse of hyperbolic trigonometric functions
- P., Polidoro 2021:

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- asymptotically sharp as $\mathbf{h}$ goes to 0^+ and $+\infty$

Yosida's parametrix

- $\mathcal{L}$ parabolic operator on a Riemannian manifold

Yosida (1953):

$$H_1(t,x;T,y) = \frac{(T-t)^{-n/2}}{\sqrt{(2\pi)^n \det a(y)}} \exp\left(-\frac{1}{2}\frac{d^2(x,y)}{T-t}\right)$$

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$$d(x,y)^2 \to d_y^2(x,y) = \langle a^{-1}(y)(y-x), y-x \rangle$$

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Adaptation to hypoelliptic setting

- $\mathcal{L}$ Asian Black-Scholes operator

P. and Polidoro (2021):

$$H_1(t,x;T,y) := \frac{1}{2\pi\sqrt{\det \mathbf{C}(y_1, T-t)}} \exp\left(-\frac{1}{2}\Psi(t,x;T,y)\right)$$

$$\mathbf{C}(\sigma, s) := \sigma^2 \begin{pmatrix} s & -\frac{s^2}{2} \\ -\frac{s^2}{2} & \frac{s^3}{3} \end{pmatrix}$$

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- $\mathbf{C}$ yields right behavior near singularity

$$\Psi(0,x;T,y) = \\ \left\langle \mathbf{C}^{-1}(y_1,T)\big(x_1 - y_1, x_2 - y_2 + Ty_1\big), x_1 - y_1, x_2 - y_2 + Ty_1 \right\rangle \\ + o\big(|x_1 - y_1|^2 + |x_2 - y_2 + Ty_1|^2\big)$$

Why does it work?

- Setting $z = (t, x)$, $w = (T, y)$
$$\mathcal{L}H_1(z;w) = \left(\left(\frac{x_1}{2}\partial_{x_1}\Psi(z;w)\right)^2 - Y\Psi(z;w) + f(z,w)\right)H_1(z;w),$$

with
$$f(t,x;T,y) := \frac{2}{T-t} - \frac{x_1^2\partial_{x_1x_1}\Psi(t,x;T,y)}{4}.$$

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Singular terms:

- $f(z, w)$ singular of order $\frac{1}{T-t}$
- functions

$$\left(\frac{x_1}{2} \partial_{x_1} \Psi(z; w) \right)^2 \quad \text{and} \quad Y\Psi(z; w)$$

  singular of order $\frac{1}{(T-t)^2}$

HJB equation

Lemma

$$Y\Psi(t,x;T,y) = \left(\frac{x_1}{2}\partial_{x_1}\Psi(t,x;T,y)\right)^2$$

Furthermore, the optimal control ω satisfies

$$\omega(s) = -\frac{\gamma_1(s)}{2}\partial_{\gamma_1}\Psi\big(s,\gamma(s);T,y\big), \qquad s \in [t,T].$$

- the proof is classic
- owes to the regularity of $\Psi(\cdot,\cdot;T,y)$ (smooth) and $\omega(s)$ (continuous)

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$$\downarrow$$

no more $\dfrac{1}{(T-t)^2}$

Setting now:

$$H := H_1 \mathbf{u},$$

$$\downarrow$$

$$\mathcal{L}H = \Big(\underbrace{\big(Y - g\partial_{x_1} + f\big)\mathbf{u}}_{=0} + \frac{x_1^2}{2}\partial_{x_1 x_1}\mathbf{u} \Big) H_1$$

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$$g(t,x;T,y) := \frac{x_1^2}{2}\partial_{x_1}\Psi(t,x;T,y)$$

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Lemma

For any (T,y), the integral curves of the vector field

$$(t,x) \mapsto Y - g(t,x;T,y)\partial_{x_1}, \qquad t < T, \ x_2 < y_2$$

are the optimal curves $\big(s, \gamma(s)\big)$

Partial proof!

At some point we need the bound

$$\int_{\mathbb{R}^+}\int H_1(t,x;s,\xi)H_1(s,\xi;T,y)\mathrm{d}\xi \le C_{T-t}H_1(t,x;T,y), \quad s\in]t,T[$$

uniformly in $x,y\in]0,+\infty[\times\mathbb{R}$ with $x_2<y_2$

- So far strong numerical evidence

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- Standard techniques do not work (probability, PDEs, geometry)

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- So far strong numerical evidence
- In the parabolic setting: Chapman-Kolmogorov equation (semigroup property)
- Standard techniques do not work (probability, PDEs, geometry)
- A key identity (Euclidean case):

$$\Psi(t,x;s,\xi)+\Psi(s,\xi;T,y)=\Psi(t,x;T,y)+\Psi\left(0,\gamma(s);\frac{(T-s)(s-t)}{T-t},\xi\right)$$

Future developments

- proof of C. K. inequality (perhaps localization)

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- McKean-Vlasov SDEs and SPDEs

Bibliography I

Aronson, D. G. (1967). Bounds for the fundamental solution of a parabolic equation. *Bulletin of the American Mathematical society 73*(6), 890–896.

Azencott, R. (1984). Densité des diffusions en temps petit: développements asymptotiques. I. In *Seminar on probability, XVIII*, Volume 1059 of *Lecture Notes in Math.*, pp. 402–498. Springer, Berlin.

Cibelli, G., S. Polidoro, and F. Rossi (2019). Sharp estimates for geman–yor processes and applications to arithmetic average asian options. *Journal de Mathématiques Pures et Appliquées 129*, 87–130.

Da Prato, G. and A. Lunardi (1995). On the ornstein-uhlenbeck operator in spaces of continuous functions. *Journal of Functional Analysis 131*(1), 94–114.

Bibliography II

Di Francesco, M. and A. Pascucci (2005). On a class of degenerate parabolic equations of Kolmogorov type. *AMRX Appl. Math. Res. Express* (3), 77–116.

Friedman, A. (1964). *Partial differential equations of parabolic type.* Englewood Cliffs, N.J.: Prentice-Hall Inc.

Friz, P. K., J. Gatheral, A. Gulisashvili, A. Jacquier, and J. Teichmann (2015). *Large deviations and asymptotic methods in finance*, Volume 110. Springer.

Gatheral, J., E. P. Hsu, P. Laurence, C. Ouyang, and T.-H. Wang (2012). Asymptotics of implied volatility in local volatility models. *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics 22*(4), 591–620.

Imbert, C. and L. Silvestre (2021). The schauder estimate for kinetic integral equations. *Analysis & PDE 14*(1), 171–204.

Bibliography III

Lanconelli, A., A. Pascucci, and S. Polidoro (2020). Gaussian lower bounds for non-homogeneous kolmogorov equations with measurable coefficients. *Journal of Evolution Equations 20*(4), 1399–1417.

Léandre, R. (1987a). Majoration en temps petit de la densité d'une diffusion dégénérée. *Probability theory and related fields 74*(2), 289–294.

Léandre, R. (1987b). Minoration en temps petit de la densité d'une diffusion dégénérée. *Journal of Functional Analysis 74*(2), 399–414.

Levi, E. E. (1907). Sulle equazioni lineari totalmente ellittiche alle derivate parziali. *Rend. Circ. Mat. Palermo 24*, 275–317.

Manfredini, M. (1997). The dirichlet problem for a class of ultraparabolic equations. *Advances in Differential Equations 2*(5), 831–866.

Bibliography IV

Menozzi, S. (2011). Parametrix techniques and martingale problems for some degenerate kolmogorov equations. *Electronic communications in probability 16*, 234–250.

Molchanov, S. A. (1975). Diffusion processes and riemannian geometry. *Russian Mathematical Surveys 30*(1), 1.

P., S., A. Pascucci, and M. Pignotti (2016). Intrinsic taylor formula for kolmogorov-type homogeneous groups. *Journal of Mathematical Analysis and Applications 435*(2), 1054–1087.

P., S. and S. Polidoro (2021). A Yosida's parametrix approach to Varadhan?s estimates for a degenerate diffusion under the weak Hörmander condition. *ArXiv*.

Polidoro, S. (1994). On a class of ultraparabolic operators of Kolmogorov-Fokker-Planck type. *Matematiche (Catania) 49*(1), 53–105 (1995).

Bibliography V

Varadhan, S. R. (1967a). Diffusion processes in a small time interval. *Communications on Pure and Applied Mathematics 20*(4), 659–685.

Varadhan, S. R. S. (1967b). On the behavior of the fundamental solution of the heat equation with variable coefficients. *Communications on Pure and Applied Mathematics 20*(2), 431–455.

Yor, M. (1992). On some exponential functionals of Brownian motion. *Adv. in Appl. Probab. 24*(3), 509–531.

Yosida, K. (1953). On the fundamental solution of the parabolic equation in a riemannian space. *Osaka Mathematical Journal 5*(1), 65–74.

THANK YOU
FOR YOUR ATTENTION!