Mean value formulae for degenerate parabolic operators

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Kolmogorov Operators and their Applications

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The Mean Value Formula for harmonic functions

Fundamental solution ($N \geq 3$):

$$\Gamma(x, x_0) = \frac{1}{N(2-N)\omega_N |x - x_0|^{N-2}}$$

Green's formula (u harmonic):

$$\begin{aligned} 0 &= \int_{B_r(x_0)\setminus B_\varepsilon(x_0)} u\Delta\Gamma - \Gamma\Delta u \\ &= \frac{1}{N\omega_N r^{N-1}} \int_{\partial B_r(x_0)} u \qquad \textit{mean value} \\ &\quad - \frac{1}{N\omega_N \varepsilon^{N-1}} \int_{\partial B_\varepsilon(x_0)} u \qquad \to -u(x_0) \\ &\quad \frac{1}{N(2-N)\omega_N \varepsilon^{N-2}} \int_{\partial B_\varepsilon(x_0)} \partial_\nu u \quad \to 0 \end{aligned}$$

The Mean Value Formula for caloric functions

$Hu = \Delta u - \partial_t u = 0$, $z = (x, t)$, $z_0 = (x_0, t_0)$,

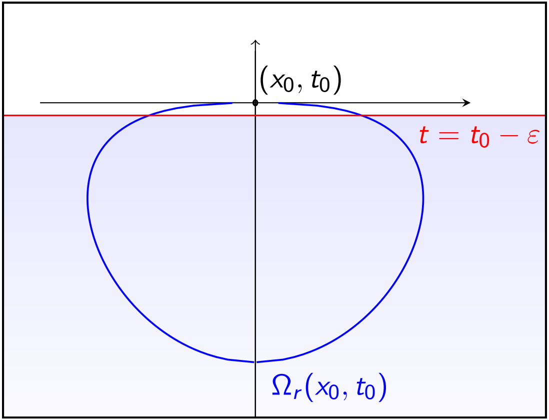
$$\Gamma(z_0, z) = \frac{1}{(4\pi(t - t_0))^{N/2}} \exp\Big\{ \frac{|x - x_0|^2}{4(t - t_0)} \Big\}$$

$$\Omega_r(z_0) = \Big\{ z \in \mathbb{R}^{N+1} \mid \Gamma(z_0; z) > \tfrac{1}{r^N} \Big\}, \psi_r(z_0) = \partial\Omega_r(z_0)$$

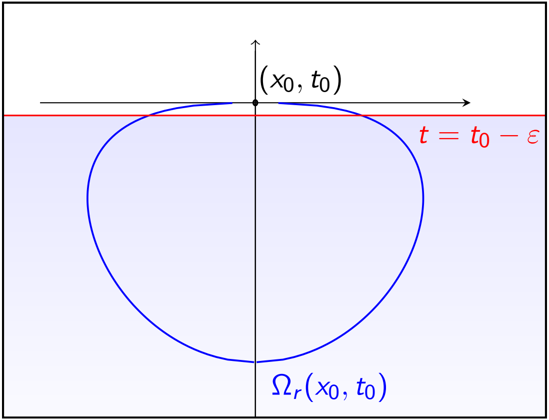
BUT $z_0 \in \psi_r(z_0)$. Let $v = \Gamma - r^{-N}$. As $\varepsilon \to 0$:

$$\begin{aligned} 0 &= \int_{\Omega_r(z_0)\cap\{t<t_0-\varepsilon\}} u \ Hv - v \ Hu \\ &= \int_{\psi_r(z_0)\cap\{t<t_0-\varepsilon\}} u \ \partial_\nu v \qquad \to \int_{\psi_r(z_0)} u \ \frac{|\nabla_x \Gamma|^2}{|\nabla_{(x,t)}\Gamma|} \\ &\quad - \int_{\Omega_r(z_0)\cap\{t=t_0-\varepsilon\}} u \ \partial_\nu v \quad \to -u(z_0) \\ &\quad + \int_{\Omega_r(z_0)\cap\{t=t_0-\varepsilon\}} \partial_\nu u \ v \quad \to 0 \end{aligned}$$

Domain of integration



Domain of integration



Thank you, Sergio!

General uniformly parabolic operators

$$\mathscr{L} u := \mathrm{div}_x(A(z)\nabla_x u) + \langle b(z), \nabla_x u\rangle + c(z)u - \frac{\partial u}{\partial t}$$

$A = (a_{ij})$ uniformly elliptic

Basic ingredients:

1. A fundamental solution Γ
2. An integration by parts on the (truncated) level sets of Γ
3. A Green formula

Applications:

1. Maximum principles
2. Harnack inequalities

Notation

Fundamental solution $\Gamma(\cdot;\zeta) \in C^{2,1}(\mathbb{R}^N \times]\tau,\infty[)$,
$\mathscr{L}\,\Gamma(\cdot;\zeta) = 0$ in $\mathbb{R}^N \times]\tau,\infty[$; $u(z) = \int_{\mathbb{R}^N} \Gamma(z;\xi,\tau)\phi(\xi)d\xi$ solves

$$\begin{cases} \mathscr{L}\,u = 0 \\ u(\cdot,\tau) = \varphi \in C_c(\mathbb{R}^N) \end{cases}$$

$z_0 = (x_0, t_0) \in \mathbb{R}^{N+1}$, $r > 0$: Define

$$\Omega_r(z_0) := \left\{ z \in \mathbb{R}^{N+1} \mid \Gamma(z_0;z) > \tfrac{1}{r^N} \right\}$$
$$\psi_r(z_0) := \partial\Omega_r(z_0)$$

$$K(z_0;z) := \frac{\langle A(z)\nabla_x\Gamma(z_0;z), \nabla_x\Gamma(z_0;z)\rangle}{|\nabla_{(x,t)}\Gamma(z_0;z)|},$$
$$M(z_0;z) := \frac{\langle A(z)\nabla_x\Gamma(z_0;z), \nabla_x\Gamma(z_0;z)\rangle}{\Gamma(z_0;z)^2}.$$

The Mean Value Formula for parabolic operators with smooth coefficients

Theorem (Fabes–Garofalo) Assume $a_{ij}, b_i, c, f \in C^\infty$, $\mathscr{L} u = f$. Then $\Gamma(z,\zeta)$ is C^∞ in $\{z \neq \zeta\}$ and

$$
\begin{aligned}
u(z_0) = & \int_{\psi_r(z_0)} K(z_0,z)u(z)\ d\sigma && (1)\\
& + \int_{\Omega_r(z_0)} f(r^{-N} - \Gamma) + \big(\mathrm{div} b - c\big) u dz \\
= & r^{-N} \int_{\Omega_r(z_0)} M(z_0;z)u(z)dz && (2)\\
& + \frac{N}{r^N}\int_0^r \varrho^{N-1} d\varrho \int_{\Omega_\varrho(z_0)} f(r^{-N} - \Gamma) + \big(\mathrm{div} b - c\big) u dz
\end{aligned}
$$

Proof

$\Omega \subset \mathbb{R}^{N+1}$, $\mathscr{L} u = f$ in Ω, $z_0 = (x_0, t_0) \in \Omega$,
$v(z) = \Gamma(z_0, z) - r^{-N}$.

$$
\begin{aligned}
u\mathscr{L}^* v - v\mathscr{L} u &= r^{-N}(\mathrm{div} b - c)u - vf \\
&= \mathrm{div}_x\big(uA\nabla_x v - vA\nabla_x u\big) - \mathrm{div}_x\big(uvb\big) + \partial_t(uv) \\
&= \mathrm{div}_{(x,t)}\Phi,
\end{aligned}
$$

where $\Phi = \Big(uA\nabla_x v - vA\nabla_x u - uvb, uv\Big)$.
Integrating on $\Omega_r(z_0) \cap \{t < t_0 - \varepsilon\}$, using the divergence theorem and letting $\varepsilon \to 0$

Proof continued

$$\int_{\Omega_r(z_0)\cap\{t<t_0-\varepsilon\}} \Big(r^{-N}(\mathrm{div}\, b - c)u - vf \Big) dz =$$

$$- \int_{\psi_r(z_0)\cap\{t<t_0-\varepsilon\}} \langle \nu, \Phi \rangle dH^N \quad \to - \int_{\psi_r(z_0)} K(z_0, z) u dH^N$$

$$+ \int_{\Omega_r(z_0)\cap\{t=t_0-\varepsilon\}} \langle e, \Phi \rangle dH^N \quad \to u(z_0)$$

where $\nu(z) = \frac{\nabla_{(x,t)}\Gamma(z_0,z)}{|\nabla_{(x,t)}\Gamma(z_0,z)|}$, $e = (0, \dots, 0, 1)$, and (1) follows.
To get (2), write $u(z_0) = \frac{N}{r^N} \int_0^r \varrho^{N-1} u(z_0) d\varrho$ and notice that

$$\int_0^r \varrho^{N-1} \int_{\psi_\varrho(z_0)} K(z_0; z) u dH^N \, d\varrho$$

$$= \frac{1}{N} \int_0^r \varrho^{N-1} \int_{\Omega_\varrho(z_0)} M(z_0; z) u dz$$

Parabolic operators with C^1 coefficients

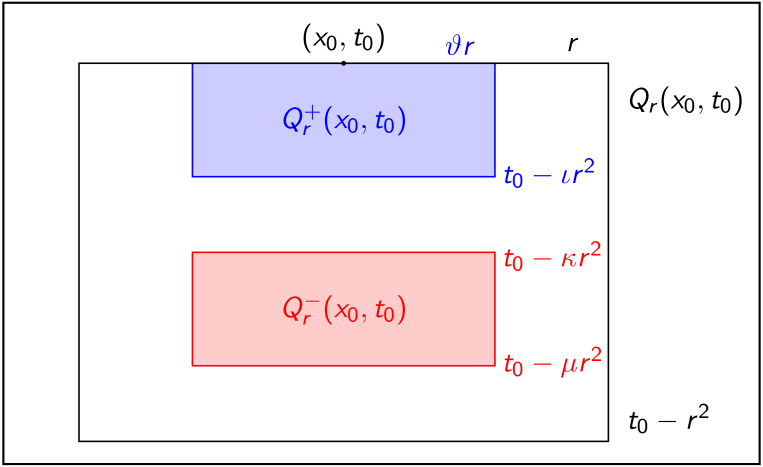
Assume $a_{ij}, \partial_{x_i} a_{ij}, b, \partial_{x_i} b_i, c \in C^{\alpha,\alpha/2}$
Difficulties: Γ is only C^1, hence (Dubovicki) a.a. its level sets have a almost C^1 boundary, i.e., there is a closed set $M \subset \partial\{\Gamma > c\}$ such that $\partial\{\Gamma > c\} \setminus M$ is C^1 for a.e. $c \in \mathbb{R}$.
How to prove (1) and (2):
- use a refinement of the classical divergence theorem on sets with almost C^1 boundary
- use the theory of sets with finite perimeter

Theorem (Malagoli-P-Polidoro) Under the above hypotheses, (1) and (2) hold for a.e. $r > 0$.

Applications: A Harnack inequality

$Q_r(z_0) = B_r(x_0) \times]t_0 - r^2, t_0[$, $0 < \iota < \kappa < \mu < 1$, $0 < \vartheta < 1$:

$Q_r^-(z_0) = B_{\vartheta r}(x_0) \times]t_0 - \kappa r^2, t_0 - \mu r^2[$, $Q_r^+(z_0) = B_{\vartheta r}(x_0) \times]t_0 - \iota r^2, t_0[$.



Thanks again, Sergio!

Theorem (Malagoli-P-Polidoro)

Choose positive constants R_0 and $\iota, \kappa, \mu, \vartheta$ as above and let Ω be an open subset of $\mathbb{R}^{N+1}$. Then there exists a positive constant C_H, only depending on $\mathscr{L}$, on R_0 and on the constant that define the cylinders Q, Q^+, Q^-, such that the following inequality holds. For every $z_0 \in \Omega$ and for every positive r such that $r \le R_0$ and $Q_r(z_0) \subset \Omega$ we have

$$\sup_{Q_r^-(z_0)} u \le C_H \inf_{Q_r^+(z_0)} u$$

for every $u \ge 0$ solution to $\mathscr{L} u = 0$ in Ω.

Degenerate parabolic operators

$$X_j(x) = \sum_{k=1}^{N} \varphi_k^j(x)\partial_{x_k},\ j = 0, \dots, m,\ \varphi_k^j \in C^\infty, X_{m+1} = X_0 - \partial_t,$$

$$\mathscr{L} u = \sum_{i,j=1}^{m} X_i \left(a_{ij} X_j u\right) + X_0 u + \sum_{j=1}^{m} b_j X_j u + cu - \partial_t u$$

$A = (a_{ij})$ uniformly elliptic. Assume (Hörmander condition)

[$H1$] $\mathfrak{g} = \mathrm{Lie}(X_1, \dots, X_m, X_{m+1}),\ \mathrm{rank}\,\mathfrak{g}(z) = N + 1\ \forall z.$

[$H2$] $\exists$ a homogeneous Lie group $\mathbb{G} = (R^{N+1}, \circ, \delta_\lambda)$ s.t.

(i) $X_1, \dots, X_m, X_{m+1}$ are left translation invariant on $\mathbb{G}$;

(ii) $X_1, \dots, X_m$ are δ_λ-homogeneous of degree one and X_{m+1} is δ_λ-homogeneous of degree two;

[$H3$] $\exists$ a fundamental solution Γ^* for the adjoint operator $\mathscr{L}^*$.

The group structure

$\mathbb{G} = \left(R^{N+1}, \circ\right)$ is *homogeneous* if

$$\delta_\lambda(z \circ \zeta) = (\delta_\lambda z) \circ (\delta_\lambda \zeta)\,, \quad \textit{forall } z, \zeta \in \mathbb{R}^{N+1} \textit{ and } \lambda > 0.$$

$$[H1],\ [H2] \quad \Longrightarrow \quad \mathfrak{g} = V_1 \oplus \cdots \oplus V_\nu,$$

$$\begin{aligned}
V_1 =& \text{span}\left\{X_1, \dots X_m\right\}, \\
V_2 =& \text{span}\left\{X_{m+1}, [X_i, X_j], i, j = 1, \dots, m\right\}, \\
V_k =& \text{span}\left\{[X_i, X_j] \mid X_i \in V_i, X_j \in V_j, i + j = k\right\}, \qquad k = 3, \dots, \nu.
\end{aligned}$$

$\dim V_j = n_j$, δ_λ is given by $\delta_\lambda = \mathrm{diag}(\lambda \mathbb{I}_{n_1}, \lambda^2 \mathbb{I}_{n_2}, \dots, \lambda^\nu \mathbb{I}_{n_\nu})$,
$Q + 2 = n_1 + \cdots + n_\nu$ is called *homogeneous dimension* of $\mathbb{G}$,
$\det \delta_\lambda = \lambda^{Q+2}$.

Example 1: Heat equation in the Heisenberg group

$\mathbb{H}^n = (\mathbb{R}^{2n+1}, \cdot)$,
$(x, y, s) \cdot (x', y', s') = \Big(x + x', y + y', s + s' + 2\sum_{j=1}^n (x_j' y_j - x_j y_j')\Big)$
$\tilde{\delta}_\lambda(x, y, s) = \big(\lambda x, \lambda y, \lambda^2 s\big) \quad \mathbb{R}^{2n+1} = V_1 \oplus V_2$,
$V_1 = \{(x, y, 0) \mid x, y, \in \mathbb{R}^n\}$ $V_2 = \{= (0, 0, s) \mid s \in \mathbb{R}\}$.
$X_j = \partial_{x_j} + 2y_j\partial_s, Y_j = \partial_{y_j} - 2x_j\partial_s, \quad j = 1, \dots, n$,
are left-invariant and $[X_j, Y_j] = -4\partial_s, \quad j = 1, \dots, n$,
sub-Laplacian $\Delta_{\mathbb{H}^n} := \sum_{j=1}^n \big(X_j^2 + Y_j^2\big)$
$\mathbb{G} = \big(\mathbb{R}^{2n+2}, \circ, (\delta_\lambda)_{\lambda>0}\big)$
$(x, y, s, t) \circ (x', y', s', t') = \Big((x, y, s) \cdot (x' y', s'), t + t'\Big)$
$\delta_\lambda(x, y, s, t) = \big(\tilde{\delta}_\lambda(x, y, s), \lambda^2 t\big)$
heat equation: $\mathscr{H} = \sum_{j=1}^n \big(X_j^2 + Y_j^2\big) - \partial_t$

Example 2: Kolmogorov operators

$$\mathscr{K} = \sum_{j=1}^{n} \partial_{x_j}^2 + \sum_{j=1}^{n} x_j \partial_{y_j} - \partial_t, \quad (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$$

$m = n$, $X_j = \partial_{x_j}$, $j = 1, \ldots, n$,

$X_{n+1} = \sum_{j=1}^{n} x_j \partial_{y_j} - \partial_t$

$\mathbb{K}\left(\mathbb{R}^{2n+1}, \circ, (\delta_\lambda)_{\lambda>0}\right)$

$(x, y, t) \circ (x' y', t') = (x + x', y + y' - t'x, t + t')$

$\delta_\lambda(x, y, t) = (\lambda x, \lambda^3 y, \lambda^2 t)$

$\mathbb{R}^{2n+1} = V_1 \oplus V_2 \oplus V_3$ with $V_1 = \{(x, 0, 0) \mid x \in \mathbb{R}^n\}$,

$V_2 = \{= (0, 0, t) \mid t \in \mathbb{R}\}$
$V_3 = \{(0, y, 0) \mid y \in \mathbb{R}^n\}$.

Distance

$\exists \varepsilon_j \in]0,1], \; j = 1, \ldots, \nu, \varepsilon_1 = 1,$

$$\textit{norm}: \quad x \mapsto \|x\|_\infty = \max_{j=1,\ldots,\nu} \{\varepsilon_j |x_j|^{1/j}\},$$

$x_j \in \mathbb{R}^{n_j}$, $|\cdot|$ the Euclidean norm

$$\textit{distance} \quad d_\infty(x,y) = \|y^{-1} \circ x\|_\infty.$$

$\forall K \subset \mathbb{R}^N$ compact $\exists c_K^-, c_K^+$

$$c_K^- |x-y| \le d_\infty(x,y) \le c_K^+ |x-y|^{\frac{1}{\nu}}, \qquad \text{for all } x, y \in K.$$

Invariance

$$d_\infty(y \circ x, y \circ z) = d_\infty(x,z), \quad d_\infty(\delta_\lambda x, \delta_\lambda y) = \lambda d_\infty(x,y),$$

Spherical Hausdorff measure

$$0 < h \leq Q + 2, \quad \mathrm{diam}_{\mathbb{G}}(E) = \sup_{x,y \in E} d_\infty(x,y)$$

$$\mathcal{S}^h_{\mathbb{G},r}(E) = \inf \left\{ \sum_{i=0}^{\infty} 2^{-h} \mathrm{diam}_{\mathbb{G}}\left(B_i\right)^h : B_i \ d_\infty\text{-balls}, \right.$$

$$\left. E \subset \bigcup_{i=0}^{\infty} B_i, \ \mathrm{diam}_{\mathbb{G}}\left(B_i\right) \leq r \right\}$$

$$\mathcal{S}^h_{\mathbb{G}}(E) = \lim_{r \downarrow 0} \mathcal{S}^h_{\mathbb{G},r}(E)$$

Derivatives and Hölder continuity

For $x_0 \in \mathbb{R}^{N+1},\ j = 1, \dots, m$, let γ be the solution of

$$\gamma'(s) = X_j(\gamma(s)), \qquad \gamma(0) = x_0.$$

The Lie derivative $X_j u(x_0)$ of u at x_0 is

$$X_j u(x_0) := \frac{d}{ds} u(\gamma(s))_{|s=0}.$$

$u \in C^{\alpha}_{\mathbb{G}}(\Omega)$ if

$$|u(x) - u(y)| \le M d_{\infty}(x, y)^{\alpha}, \quad \text{for every } x, y \in \Omega.$$

$u \in C^{1+\alpha}_{\mathbb{G}}(\Omega)$ (resp. $C^{2+\alpha}_{\mathbb{G}}(\Omega)$) if u, the derivatives $X_1 u, \dots, X_m u$ (resp. and $X_i X_j u$, $i, j = 1, \dots, m$) belong to the space $C^{\alpha}_{\mathbb{G}}(\Omega)$.

The divergence operator

$$\mathrm{div}_{\mathbb{G}} g = \sum_{i=1}^{m+1} X_i g_i, \quad g \in C^1_0\left(\mathbb{R}^{N+1}, \mathbb{R}^{m+1}\right).$$

$\mathbb{G}$–perimeters

$f \in L^1(\mathbb{R}^{N+1})$ is in $BV(\mathbb{G})$ if

$$|D_X f|\left(\mathbb{R}^{N+1}\right) = \sup\left\{\int_{\mathbb{R}^{N+1}} f(z)\mathrm{div}_{\mathbb{G}} g(z)dz : \right.$$
$$\left. g \in C_0^1\left(\mathbb{R}^{N+1}, \mathbb{R}^{m+1}\right), \|g\|_\infty \leq 1\right\}.$$

If $D_X\chi_E(\mathbb{R}^{N+1}) < \infty$ then E is a set with finite perimeter.
$f \in BV \;\Rightarrow\; D_X f$ is an $\mathbb{R}^{m+1}$-valued measure and
$D_X f = \sigma_f |D_X f|$. $f = \chi_E \;\Rightarrow\; \sigma_f = \nu_E$ (generalised inner normal).
Then (divergence theorem, weak form)

$$\int_{\mathbb{R}^{N+1}} \chi_E(z)\mathrm{div}_{\mathbb{G}} g(z)dz = -\int_{\mathbb{R}^{N+1}} \langle g, \nu_E\rangle d|D_X\chi_E|$$

The divergence theorem, revisited (Ambrosio)

Essential boundary $z \in \partial^*_{\mathbb{G}} E$ if

$$\limsup_{r \to 0} \frac{\lambda_{N+1}(E \cap B(z,r))}{\lambda_{N+1}(B(z,r))} > 0, \quad \limsup_{r \to 0} \frac{\lambda_{N+1}(B(z,r) \setminus E)}{\lambda_{N+1}(B(z,r))} > 0$$

Divergence theorem $\exists \beta_E : \mathbb{G} \to [\ell_{\mathbb{G}}, L_{\mathbb{G}}]$ s.t.

$$\int_E \mathrm{div}_{\mathbb{G}} g(z) dz = - \int_{\partial^*_{\mathbb{G}} E} \langle g, \nu_E \rangle \, \beta_E(z) \, d\mathcal{S}^{Q+1}_{\mathbb{G}}.$$

Regular surfaces (Magnani)

$S \subset \mathbb{R}^{N+1}$ is $\mathbb{G}$-regular if for any $p \in S$ there are an open neighbourhood U of p and $f \in C^1_{\mathbb{G}}(U)$ such that

$$S \cap U = \{f = 0,\ \nabla_{\mathbb{G}} f(z) \neq 0\},\ \nu_E(z) = -\frac{\nabla_{\mathbb{G}} f(z)}{|\nabla_{\mathbb{G}} f(z)|},$$

Then $\partial^*_{\mathbb{G}}(E \cap U) = \partial(E \cap U)$. $\nu_S^{\perp}(z) =$ the codimension 1 subspace of V_1 orthogonal to $\nu(z)$, $N(z) = \nu_E^{\perp}(z) \oplus V_2 \oplus \cdots \oplus V_\nu$ and

$$\beta_E(z) = \max_{w \in B(0,1)} \{H^N_{eucl}(B(w,1) \cap N(z)\} = \theta\ (constant!)$$

Remark: The above formula for β_E holds locally on the regular portions of the boundary.

The MVF for degenerate parabolic operators

Define the kernels

$$K(z_0; z) := \frac{\langle A(z)\nabla_X \Gamma(z_0; z), \nabla_X \Gamma(z_0; z)\rangle}{|\nabla_{\mathbb{G}} \Gamma(z_0; z)|},$$

$$M(z_0; z) := \frac{\langle A(z)\nabla_X \Gamma(z_0; z), \nabla_X \Gamma(z_0; z)\rangle}{\Gamma(z_0; z)^2}.$$

Theorem (P-Polidoro) Assume $a_{ij}, X_i a_{ij}, b_i, c, f \in C^\alpha$, $\mathscr{L} u = f$. Then $\Gamma(z, \zeta)$ is $C^1_{\mathbb{G}}$ in $\{z \neq \zeta\}$ and

$$u(z_0) = \int_{\psi_r(z_0)} K(z_0, z) u(z) d\mathcal{S}^{Q+1}_{\mathbb{G}} \tag{3}$$

$$+ \int_{\Omega_r(z_0)} f(r^{-(Q+2)} - \Gamma) + \big(\operatorname{div}_X b - c\big) u dz$$

$$= r^{-(Q+2)} \int_{\Omega_r(z_0)} M(z_0; z) u(z) dz \tag{4}$$

$$+ \frac{Q+2}{r^{Q+2}} \int_0^r \varrho^{Q+1} d\varrho \int_{\Omega_\varrho(z_0)} f(r^{-Q+2} - \Gamma) + \big(\operatorname{div}_X b - c\big) u dz$$

Proof

$\Omega \subset \mathbb{R}^{N+1}$, $\mathscr{L}u = f$ in Ω, $z_0 = (x_0, t_0) \in \Omega$,
$v(z) = \Gamma(z_0, z) - r^{-(Q+2)}$.

$$\begin{aligned} u\mathscr{L}^* v - v\mathscr{L}u &= r^{-(Q+2)}(\mathrm{div}_X b - c)u - vf \\ &= \mathrm{div}_X\big(uA\nabla_X v - vA\nabla_X u\big) - \mathrm{div}_X\big(uvb\big) + X_{m+1}(uv) \\ &= \mathrm{div}_{\mathbb{G}}\Phi, \end{aligned}$$

where $\Phi = \Big(uA\nabla_X v - vA\nabla_X u - uvb, uv\Big)$.
Integrating on $\Omega_r(z_0) \cap \{t < t_0 - \varepsilon\}$ and using the divergence theorem (in which form?) and letting $\varepsilon \to 0$

Proof continued

$$\int_{\Omega_r(z_0)\cap\{t<t_0-\varepsilon\}} \Big(r^{-(Q+2)}(\mathrm{div}_X\, b - c)u - vf \Big) dz =$$

$$- \int_{\mathbb{R}^{N+1}} \langle \nu_{\Omega_r(z_0)\cap\{t<t_0-\varepsilon\}}, \Phi \rangle d|D_X \chi_{\Omega_r(z_0)\cap\{t<t_0-\varepsilon\}}| \quad \to ???$$

$$+ \int_{\Omega_r(z_0)\cap\{t=t_0-\varepsilon\}} \langle e, \Phi \rangle dz \quad \to u(z_0)$$

$e = (0, \dots, 0, 1)$.

What about the “surface” integral?

Proof concluded

$$\int_{\mathbb{R}^{N+1}} \langle \nu_{\Omega_r(z_0)\cap\{t<t_0-\varepsilon\}}, \Phi\rangle d|D_X \chi_{\Omega_r(z_0)\cap\{t<t_0-\varepsilon\}}|$$

reduction to the essential boundary

$$= \int_{\partial^*_{\mathbb{G}}\Omega_r(z_0)\cap\{t<t_0-\varepsilon\}} \langle \nu_{\Omega_r(z_0)\cap\{t<t_0-\varepsilon\}}, \Phi\rangle \beta_{\Omega_r(z_0)} \cap \{t < t_0 - \varepsilon\} d\mathcal{S}^{Q+1}_{\mathbb{G}}$$

Φ vanishes where $\nabla_{\mathbb{G}}\Gamma = 0$ and $\psi_r(z_0)$ is regular elsewhere

$$= \int_{\psi_r(z_0)\cap\{t<t_0-\varepsilon\}\cap\{\nabla_{\mathbb{G}}\Gamma(z_0,\cdot)\neq 0\}} \langle \nu_{\Omega_r(z_0)\cap\{t<t_0-\varepsilon\}}, \Phi\rangle \theta \; d\mathcal{S}^{Q+1}_{\mathbb{G}}$$

as $\varepsilon \to 0$

$$\to \int_{\psi_r(z_0)\cap\{\nabla_{\mathbb{G}}\Gamma(z_0,\cdot)\neq 0\}} K(z_0, z) \; \theta \; d\mathcal{S}^{Q+1}_{\mathbb{G}}$$

Proof concluded

$$\int_{\mathbb{R}^{N+1}} \langle \nu_{\Omega_r(z_0)\cap\{t<t_0-\varepsilon\}}, \Phi\rangle d|D_X \chi_{\Omega_r(z_0)\cap\{t<t_0-\varepsilon\}}|$$

reduction to the essential boundary

$$= \int_{\partial^*_{\mathbb{G}}\Omega_r(z_0)\cap\{t<t_0-\varepsilon\}} \langle \nu_{\Omega_r(z_0)\cap\{t<t_0-\varepsilon\}}, \Phi\rangle \beta_{\Omega_r(z_0)} \cap \{t < t_0 - \varepsilon\} d\mathcal{S}^{Q+1}_{\mathbb{G}}$$

Φ vanishes where $\nabla_{\mathbb{G}}\Gamma = 0$ and $\psi_r(z_0)$ is regular elsewhere

$$= \int_{\psi_r(z_0)\cap\{t<t_0-\varepsilon\}\cap\{\nabla_{\mathbb{G}}\Gamma(z_0,\cdot)\neq 0\}} \langle \nu_{\Omega_r(z_0)\cap\{t<t_0-\varepsilon\}}, \Phi\rangle \theta \; d\mathcal{S}^{Q+1}_{\mathbb{G}}$$

as $\varepsilon \to 0$

$$\to \int_{\psi_r(z_0)\cap\{\nabla_{\mathbb{G}}\Gamma(z_0,\cdot)\neq 0\}} K(z_0, z) \; \theta \; d\mathcal{S}^{Q+1}_{\mathbb{G}}$$

To get (4), argue as in the euclidean case.

Degenerate elliptic operators

$$\mathscr{L}u := \mathrm{div}_X(A(z)\nabla_X u) + \langle b(z), \nabla_X u\rangle + c(z)u$$

with Hölder continuous coefficients.
We need $\Gamma^* =$ fund. sol. of $\mathscr{L}^*$, which is known (Bonfiglioli-Lanconelli-Uguzzoni) only for $\mathscr{L}^* = \sum_{i,j=1}^m a_{ij}X_iX_j$.
Therefore,

$$b_i = 2\sum_{j=1}^m X_j a_{ij}, \qquad c = \sum_{i,j=1}^m X_iX_j a_{ij}.$$

The group $\mathbb{G}$ is analogous, without X_{m+1}, $\dim \mathbb{G} = Q$. Kernels:

$$K_{\mathbb{G}}(x_0, x) := \theta \frac{\langle A(x)\nabla_{\mathbb{G}}\Gamma^*(x,x_0), \nabla_{\mathbb{G}}\Gamma^*(x,x_0)\rangle}{|\nabla\Gamma^*_{\mathbb{G}}(x,x_0)|},$$

$$M_{\mathbb{G}}(x_0, x) := \frac{Q}{(Q-2)}\theta \frac{\langle A(x)\nabla_{\mathbb{G}}\Gamma^*(x,x_0), \nabla_{\mathbb{G}}\Gamma^*(x,x_0)\rangle}{\Gamma^*(x,x_0)^{\frac{2(Q-1)}{Q-2}}},$$

The Mean Value Formula

Theorem (P-Polidoro) $\Omega \subset \mathbb{R}^N$, $f \in C(\Omega)$, $\mathscr{L} u = f$. For every $x_0 \in \Omega$ a.e. $r > 0$

$$u(x_0) = \int_{\psi_r(x_0)} K_{\mathbb{G}}(x_0, x) u(x)\, d\mathcal{S}_{\mathbb{G}}^{Q-1}(x) + \int_{\Omega_r(x_0)} f(x) \left(\frac{1}{r^{Q-2}} - \Gamma^*(x, x_0) \right) \, dx,$$

$$u(x_0) = \frac{1}{r^Q} \int_{\Omega_r(x_0)} M_{\mathbb{G}}(x_0, x) u(x)\, dx + \frac{Q}{r^Q} \int_0^r \left(\varrho^{Q-1} \int_{\Omega_\varrho(x_0)} f(x) \left(\frac{1}{\varrho^{Q-2}} - \Gamma^*(x, x_0) \right) dx \right) d\varrho.$$

Summary

Γ: existence, regularity $+$ estimates

$\Delta, \ \Delta - \partial_t$: explicit computations

C^∞ coeff. Γ is C^∞ hence (Sard) $\partial\{\Gamma > c\}$ is C^∞ for a.e. $c \in \mathbb{R}$

$C^{1+\alpha}$ coeff. Γ is C^1 hence (Dubovickii) $\partial\{\Gamma > c\}$ is almost-C^1 for a.e. $c \in \mathbb{R}$

degenerate case: Γ is $C^1_{\mathbb{G}}$, no regularity available hence we rely on the theory of perimeters.

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