Density and gradient estimates for Kinetic SDEs with low regularity coefficients

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Kinetic SDE

Consider the $2d$-dimensional system of SDEs

$$\begin{cases} dX_t^1 = F_1(t, X_t^1, X_t^2)dt + \sigma(t, X_t^1, X_t^2)dW_t, \\ dX_t^2 = F_2(t, X_t^1, X_t^2)dt, \end{cases} \tag{1}$$

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Applications:

- physics: Hamiltonian systems

$$H(\mathbf{x}) = V(x_2) + |x_1|^2/2 \implies F_H(\mathbf{x}) = (-\nabla_{x_2} V(x_2), x_1)^*$$

- finance: path dependent contracts

Kinetic Gaussian case

$F_1 \equiv 0$ and $F_2(t, X_t^1, X_t^2) = X_t^1$

$$dX_t^1 = dW_t, \quad dX_t^2 = X_t^1 dt, \qquad t \geq 0.$$

is a Gaussian process with mean and covariance matrix given by

$$\boldsymbol{\theta}_t(\mathbf{x}) = (x_1, x_2 + x_1 t), \qquad \mathbf{K}_t = \begin{pmatrix} t\mathbb{I}_{d\times d} & \frac{t^2}{2}\mathbb{I}_{d\times d} \\ \frac{t^2}{2}\mathbb{I}_{d\times d} & \frac{t^3}{3}\mathbb{I}_{d\times d} \end{pmatrix} > 0 \quad \forall t > 0$$

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$\Rightarrow$ the process admits a density for every $t > 0$, explicitly given by

$$\mathbf{y} \mapsto \left(\frac{\sqrt{3}}{\lambda \pi t^2}\right)^d \exp\left(-\frac{1}{2}|\mathbf{K}_t^{-\frac{1}{2}}(\mathbf{y} - \boldsymbol{\theta}_t(\mathbf{x}))|^2\right)$$

$$\mathbf{K}_t \sim \begin{pmatrix} t\mathbb{I}_{d\times d} & 0_{d\times d} \\ 0_{d\times d} & t^3\mathbb{I}_{d\times d} \end{pmatrix} =: \mathbb{T}_t.$$

- *Non-diffusive* time-scale

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- *Non-diffusive* time-scale
- Different growth-rates in the two components

$$\begin{aligned} |\nabla_{x_i} p(0,\mathbf{x};t,\mathbf{y})| &\leq \left| \left((\mathbf{K}_t^{-\frac{1}{2}} \nabla \boldsymbol{\theta}_t(\mathbf{x}))^* \mathbf{K}_t^{-\frac{1}{2}} (\boldsymbol{\theta}_t(\mathbf{x}) - \mathbf{y}) \right)_i \right| p(0,\mathbf{x};t,\mathbf{y}) \\ &\lesssim \frac{1}{t^{i-\frac{1}{2}+2d}} \exp\left(-\frac{1}{2\lambda} |\mathbb{T}_t^{-\frac{1}{2}} (\mathbf{y} - \boldsymbol{\theta}_t(\mathbf{x}))|^2 \right) \\ &=: \frac{1}{t^{i-\frac{1}{2}}} g_\lambda(t, \mathbf{y} - \boldsymbol{\theta}_t(\mathbf{x})) \end{aligned}$$

Linear drifts

- Weber (1951), Polidoro (1994), Di Francesco-Pascucci (2005), Lucertini-Pagliarani-Pascucci (2022)
- Pascucci-P. (2022)

Regular, nonlinear drift

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Regularization by noise

- Fedrizzi-Flandoli-Priola-Vovelle (2017)
- X.Zhang (2018)
- Chaudru de Raynal (2018), CdR-Honoré-Menozzi (2021-2022)

Functional framework

- *Homogeneous norm*

$$|\mathbf{x}|_{\mathbf{d}} := |x_1| + |x_2|^{\frac{1}{3}} \quad \Longrightarrow \quad |\mathbb{T}_t \mathbf{x}|_{\mathbf{d}} = t|\mathbf{x}|_{\mathbf{d}} \tag{2}$$

- Corresponding *Anisotropic Hölder spaces*: $\mathcal{C}^{j+\gamma}_{\mathbf{d}}(\mathbb{R}^{2d};\mathbb{R}^l)$

$$\|f\|_{\mathcal{C}^{j+\gamma}_{\mathbf{d}}} := \sup_{x_2\in\mathbb{R}^d} \|f(\cdot,x_2)\|_{\mathcal{C}^{j+\gamma}} + \sup_{x_1\in\mathbb{R}^d} \|f(x_1,\cdot)\|_{\mathcal{C}^{(j+\gamma)/3}} < \infty.$$

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$$f \in \mathcal{C}^{\gamma}_{\mathbf{d}} \implies |f(\mathbf{x}) - f(\mathbf{y})| \le c_\gamma \|f\|_{\mathcal{C}^{\gamma}_{\mathbf{d}}} |\mathbf{x}-\mathbf{y}|^{\gamma}_{\mathbf{d}}$$

$$f \in \mathcal{C}^{1+\gamma}_{\mathbf{d}} \implies |f(\mathbf{x}) - \underbrace{\big(f(\mathbf{y}) + \nabla_{x_1} f(\mathbf{y})(x_1 - y_1)\big)}_{=:\mathcal{T}_1 f(\mathbf{x},\mathbf{y})}| \le c_\gamma \|f\|_{\mathcal{C}^{1+\gamma}_{\mathbf{d}}} |\mathbf{x}-\mathbf{y}|^{1+\gamma}_{\mathbf{d}}$$

Assume

- For some $\gamma \in (0,1]$, $\kappa_0 \geq 1$

$$\kappa_0^{-1}|\xi| \leq \langle \sigma\sigma^*(t,\mathbf{x})\xi, \xi\rangle \leq \kappa_0|\xi|, \quad \xi \in \mathbb{R}^d$$

  and

$$\|\sigma(t,\cdot)\|_{\mathcal{C}^{\gamma}_{\mathbf{d}}(\mathbb{R}^{2d},\mathbb{R}^d)} < \kappa_0.$$

- F_1 is a measurable function with linear growth

$$|F(t,\mathbf{0})| \leq \kappa_1, \quad |F(t,\mathbf{x}) - F(t,\mathbf{y})| \leq \kappa_1(1 + |\mathbf{x}-\mathbf{y}|)$$

- For some $\gamma \in (0,1]$ and $\kappa_1, \kappa_2 > 0$, it holds that

$$\|F_2(t,\cdot)\|_{\mathcal{C}^{1+\gamma}_{\mathbf{d}}(\mathbb{R}^{2d},\mathbb{R}^d)} \leq \kappa_2.$$

  Moreover, there exists a closed convex subset $\mathcal{E} \subset GL_d(\mathbb{R})$ (the set of all invertible $d \times d$ matrices) such that $\nabla_{x_1} F_2(t,\mathbf{x}) \in \mathcal{E}$ for all $t \geq 0$ and $\mathbf{x} \in \mathbb{R}^{2d}$.

Theorem *(Chaudru de Raynal, Menozzi, P., Zhang 2022)*

$\exists!$ weak solution which admits a transition density $p(t, \mathbf{x}; s, \mathbf{y})$, $0 \le t < s \le T$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{2d}$. Moreover, $p(t, \mathbf{x}; s, \mathbf{y})$ enjoys the following estimates:

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(i) (Two sides estimates)

$$C_0^{-1} g_{\lambda_0^{-1}}\left(s-t, \boldsymbol{\theta}_{s,t}(\mathbf{x}) - \mathbf{y}\right) \le p(t, \mathbf{x}; s, \mathbf{y}) \le C_0 g_{\lambda_0}\left(s-t, \boldsymbol{\theta}_{s,t}(\mathbf{x}) - \mathbf{y}\right).$$

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(ii) (Gradient estimate in x_1)

$$|\nabla_{x_1} p(t, \mathbf{x}; s, \mathbf{y})| \lesssim_{C_1} (s-t)^{-\frac{1}{2}} g_{\lambda_1} \left(s-t, \boldsymbol{\theta}_{s,t}(\mathbf{x}) - \mathbf{y}\right).$$

(iii) (Hölder estimate in x) Let $\eta_0, \eta_1 \in (0,1)$, $j = 0, 1$

$$\left|\nabla_{x_1}^j p(t, \mathbf{x}; s, \mathbf{y}) - \nabla_{x_1}^j p(t, \mathbf{x}'; s, \mathbf{y})\right| \lesssim_{C_j} |\mathbf{x} - \mathbf{x}'|_{\mathbf{d}}^{\eta_j} (s-t)^{-\frac{j+\eta_j}{2}} \times \Big(g_{\lambda_j} \left(s-t, \boldsymbol{\theta}_{s,t}(\mathbf{x}) - \mathbf{y}\right) + g_{\lambda_j} \left(s-t, \boldsymbol{\theta}_{s,t}(\mathbf{x}') - \mathbf{y}\right) \Big).$$

About the flow:

- "$\dot{\boldsymbol{\theta}}_{s,t}(\mathbf{x}) = F(s, \boldsymbol{\theta}_{s,t}(\mathbf{x}))$" is not generally well posed.

$\boldsymbol{\theta}_{t,s}(x)$ can be replaced by any Peano flow associated solving:

$$\dot{\widetilde{\boldsymbol{\theta}}}_{s,t}(\mathbf{x}) = \widetilde{F}(s, \widetilde{\boldsymbol{\theta}}_{s,t}(\mathbf{x})), \quad \boldsymbol{\theta}_{t,t}(\mathbf{x}) = \mathbf{x},$$

where

$$\widetilde{F}(s, \mathbf{x}) = ([F_1(s, \cdot) * \rho_1](\mathbf{x}), F_2(t, \mathbf{x})).$$

Equivalently $\hat{\boldsymbol{\theta}}_{s,t}(\mathbf{x})$ associated with

$$\hat{F}(s, \mathbf{x}) = ([F_1(s, \cdot) * \rho_1](\mathbf{x}), [F_2(s, \cdot) * \rho_{|s-t|^{3/2}}](\mathbf{x})).$$

We have

$$|\mathbb{T}_{s-t}^{-1}(\widetilde{\boldsymbol{\theta}}_{s,t}(\mathbf{x}) - \hat{\boldsymbol{\theta}}_{s,t}(\mathbf{x}))| \leq C$$

Perturbative argument

- Consider a *linear* approximation of the associated Kolmogorov operator: for fixed $(\tau, \boldsymbol{\xi}) \in \mathbb{R}^+ \times \mathbb{R}^{2d}$

$$\widetilde{\mathcal{K}}_t^{\tau,\boldsymbol{\xi}} = \frac{1}{2}\mathrm{tr}\left(\sigma\sigma^*(t, \boldsymbol{\theta}_{t,\tau}(\boldsymbol{\xi}))\nabla_{x_1}^2\right) + \langle \widetilde{F}^{\tau,\boldsymbol{\xi}}(t,x), \nabla\rangle + \partial_t$$

$$\widetilde{F}^{\tau,\boldsymbol{\xi}}(t,x) := F(t, \boldsymbol{\theta}_{t,\tau}(\boldsymbol{\xi})) + (DF)\left(t, \boldsymbol{\theta}_{t,\tau}(\boldsymbol{\xi})\right)\left(\mathbf{x} - \boldsymbol{\theta}_{t,\tau}(\boldsymbol{\xi})\right),$$

where

$$DF := \begin{pmatrix} 0_{d\times d} & 0_{d\times d} \\ \nabla_{x_1} F_2 & 0_{d\times d} \end{pmatrix}.$$

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- *First step approximation* of p: $Z(t,\mathbf{x};s,\mathbf{y}) := \widetilde{p}^{(s,\mathbf{y})}(t,\mathbf{x};s,\mathbf{y})$

$$Z(t,\mathbf{x};s,\mathbf{y}) \sim g_\lambda(s-t, \boldsymbol{\theta}_{s,t}(\mathbf{x}) - \mathbf{y})$$

$$|\nabla_{x_1}^j Z(t,\mathbf{x};s,\mathbf{y})| \lesssim (s-t)^{-\frac{j}{2}} g_\lambda(s-t, \boldsymbol{\theta}_{s,t}(\mathbf{x}) - \mathbf{y})$$

$$|\nabla_{x_2} Z(t,\mathbf{x};s,\mathbf{y})| \lesssim (s-t)^{-\frac{3}{2}} g_\lambda(s-t, \boldsymbol{\theta}_{s,t}(\mathbf{x}) - \mathbf{y})$$

- First step expansion (Duhamel)

$$p = Z + p \otimes (\mathcal{K} - \widetilde{\mathcal{K}})Z$$

In general

$$\left.\begin{aligned} |x_1| &\asymp t^{\frac{1}{2}} \\ |x_2| &\asymp t^{\frac{3}{2}} \end{aligned}\right\} \implies |x|_{\mathbf{d}} \asymp t^{\frac{1}{2}}$$

Diffusion perturbation

$$\frac{|\mathbf{y} - \boldsymbol{\theta}_t(\mathbf{x})|_{\mathbf{d}}^{\gamma}}{t} \sim t^{\frac{\gamma}{2}-1}$$

Drift perturbation (second component)

$$\frac{|\mathbf{y} - \boldsymbol{\theta}_t(\mathbf{x})|_{\mathbf{d}}^{1+\gamma}}{t^{\frac{3}{2}}} \sim t^{\frac{\gamma}{2}-1}$$

Theorem *(Chaudru de Raynal, Menozzi, P., Zhang 2022)*

i) If, for some $\gamma > 0$,

$$\|F_1(t,\cdot)\|_{\mathcal{C}^{\gamma}_{\mathbf{d}}} \le \kappa_1 \quad t \ge 0$$

$\Longrightarrow$ Existence and estimates for the second order derivative

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ii) For gradient estimates in the degenerate component x_2, we need extra regularity, since for kinetic operators we only have 2/3-gain of regularity in x_2

If σ and F_1 also satisfy that

$$|\sigma(t,\mathbf{x}) - \sigma(t,\mathbf{y})| \leq \kappa_0(|x_1 - y_1| + |x_2 - y_2|^{\frac{1+\gamma}{3}})$$
$$|F_1(t,\mathbf{x}) - F_1(t,\mathbf{y})| \leq \kappa_1(|x_1 - y_1|^{\gamma} + |x_2 - y_2|^{\frac{1+\gamma}{3}}),$$

$\Longrightarrow$ Existence and estimates for the *first order derivative in* x_2

- (Gradient estimate in x_3)

$$|\nabla_{x_2} p(t, \mathbf{x}; s, \mathbf{y})| \lesssim_{C_3} (s-t)^{-\frac{3}{2}} g_{\lambda_3}\left(s-t, \boldsymbol{\theta}_{s,t}(\mathbf{x}) - \mathbf{y}\right).$$

- (Hölder estimate in x) For $\eta_3 < \gamma$

$$|\nabla_{x_2} p(t, \mathbf{x}; s, \mathbf{y}) - \nabla_{x_2} p(t, \mathbf{x}'; s, \mathbf{y})| \lesssim_{C_4} |\mathbf{x} - \mathbf{x}'|_{\mathbf{d}}^{\eta_3} (s-t)^{-\frac{3+\eta_3}{2}}$$
$$\times \Big(g_{\lambda_4}\left(s-t, \boldsymbol{\theta}_{s,t}(\mathbf{x}) - \mathbf{y}\right) + g_{\lambda_4}\left(s-t, \boldsymbol{\theta}_{s,t}(\mathbf{x}') - \mathbf{y}\right) \Big).$$

Reference

Chaudru de Raynal - Menozzi - P. - Zhang. *Heat kernel and gradient estimates for kinetic SDEs with low regularity coefficients*, **arXiv**:2203.11515 (2022)

Thank you for your attention