The Dirichlet problem for a family of totally degenerate differential operators

Joint work with M. Manfredini & S. Polidoro

Mirco Piccinini

Dipartimento di Scienze Matematiche, Fisiche e Informatiche, Università di Parma
Parco Area delle Scienze 53/a, Campus, 43124 Parma, Italy

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The main problem

Consider the Kolmogorov-type operator given by

$$\mathcal{L} = \sum_{j=1}^{m_0} \left(t^{\vartheta} \partial_{x_j}\right)^2 + \langle Bx, \nabla \rangle - \partial_t, \qquad \vartheta \in \mathbb{N},\ z = (x,t) \in \mathbb{R}^{N+1}.$$

where the matrix $B := (b_{ij})_{i,j=1,\dots,N}$ has the form

$$B = \begin{pmatrix} \mathbb{O} & \mathbb{O} & \dots & \mathbb{O} & \mathbb{O} \\ B_1 & \mathbb{O} & \dots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & B_2 & \dots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & B_\kappa & \mathbb{O} \end{pmatrix}$$

with B_j is a $m_j \times m_{j-1}$ matrix of rank m_j with $j = 1, 2, \dots, \kappa$

$$m_0 \geq m_1 \geq \ldots \geq m_\kappa \geq 1, \quad \text{and} \quad m_0 + m_1 + \ldots + m_\kappa = N.$$

The main problem

In the framework of the Potential Theory, we address

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega \subset \mathbb{R}^{N+1}, \\ u_{|\partial\Omega} = \varphi \in C_c(\partial\Omega). \end{cases} \tag{1}$$

We prove

- the existence of a generalized solution in the sense of Perron H_φ^Ω to (1);
- the generalized solution $H_\varphi^\Omega \in C^\infty$ and is a classical solution to $\mathcal{L}u = 0$;
- a characterization of boundary $\mathcal{L}$-regularity.

We say that $z_0 \in \partial\Omega$ is $\mathcal{L}$-regular if

$$H_\varphi^\Omega(z) \to \varphi(z_0) \qquad \text{as } z \to z_0, \, \forall \varphi \in C_c(\partial\Omega).$$

The underlying geomtetrical structure

Given the matrix

$$E(s) := e^{-sB}$$

it was proven by Lanconelli & Polidoro[1] that equipped with the composition law

$$(x, t) \circ (\xi, \tau) = (\xi + E(\tau)x, t + \tau) \tag{2}$$

$\mathbb{K} := (\mathbb{R}^{N+1}, \circ)$ is a non-commutative Lie group.

$\mathcal{L}$ is not translation invariant as $\vartheta \geq 1$. Nevertheless, the matrix $E(s)$ will have a key role in applying the abstract Perron method to study probelm (1).

[1] E. Lanconelli & S. Polidoro, Rend. Sem. Mat. Univ. Pol. Torino 1994

The underlying geomtetrical structure

For every $r > 0$, we denote with $\delta(r)$ the automorphism on $\mathbb{R}^{N+1}$ given by

$$\delta(r) = (\delta_0(r), r^2), \qquad \forall r > 0$$

where

$$\delta_0(r) = r^{2\vartheta} \mathrm{diag}(r\mathbb{I}_{m_0}, r^3\mathbb{I}_{m_1}, \ldots, r^{2\kappa+1}\mathbb{I}_{m_\kappa})$$

$\mathcal{L}$ is homogeneous of degree two with respect to $\delta(\cdot)$,

$$\mathcal{L} \circ \delta(r) = r^2 \delta(r) \circ \mathcal{L}, \qquad \forall\ r > 0$$

The dilation group $\delta(\cdot)$ will be the starting point in stating the Zaremba-type exterior cone condition which ensures boundary $\mathcal{L}$-regularity.

The hypoellipticity of $\mathcal{L}$

Defined the vector fields[2]

$$X_j := t^{\vartheta}\partial_{x_j},\ Y := \langle Bx, \nabla\rangle - \partial_t,$$

and rewrite $\mathcal{L}$ as follows

$$\mathcal{L} = \sum_{j=1}^{m_0} X_j^2 + Y.$$

The particular form of the matrix B is equivalent to requiring that the vector fields $\{X_1, \dots, X_{m_0}, Y\}$ satisfies the Hörmander rank condition

$$\text{rank Lie}(X_1, \dots, X_{m_0}, Y)(z) = N + 1, \qquad \forall z \in \mathbb{R}^{N+1}.$$

[2] As customary we identify any vector field X with the vector valued function whose entries are the coefficients of X

$$X = \sum_{j=1}^{N} c_j(z)\partial_{x_j} + c_0(z)\partial_t \simeq (c_1(z), \dots, c_N(z), c_0(z))$$

The fundamental solution

Hörmander's condition is equivalent to requiring that

$$C(\tau, t) := \int_{\tau}^{t} s^{2\vartheta} E(s) \begin{pmatrix} \mathbb{I}_{m_0} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix} E^T(s) ds > 0, \qquad \forall\, t > \tau.$$

Condition above is very important in our setting. Indeed, we are able to find the explicit expression of the fundamental solution of $\mathcal{L}$

$$\Gamma(z; z_0) = \begin{cases} \frac{(4\pi)^{-N/2}}{\sqrt{\det C(t_0,t)}} e^{-\frac{1}{4}\langle C^{-1}(t_0,t)\left(x - E(t-t_0)x_0\right), x - E(t-t_0)x_0\rangle} & \text{if } t > t_0, \\ 0 & \text{if } t \leq t_0. \end{cases}$$

The explicit expression of Γ will be employed in the proof of an equivalent characterization of boundary $\mathcal{L}$-regularity.

Perron's Method: resolutive and regular sets

Perron's Method works assuming that there exists a basis for the topology of $\mathbb{R}^{N+1}$ of resolutive sets $\{U_\imath\}_\imath$.

We call *resolutive* any open set U for which there exists a generalized Perron solution H_φ^U, for any $\varphi \in C_c(\partial U)$.

It is not required that H_φ^U attains the boundary value $\varphi(z_0)$ in any point $z_0 \in \partial U$.

A resolutive set U is said regular if any $z_0 \in \partial U$ is $\mathcal{L}$-regular.

The development of the Potential Theory is simpler in the case we assume the existence of a basis of regular sets.

Perron's Method: the work of J. M. Bony

Bony[3] considered the boundary value problem for degenerate operators in the form

$$\mathcal{L} = \sum_{j=1}^{m} X_j^2 + Y,$$

- $X_1, \dots, X_m, Y \in C^\infty(\Omega)$;
- $\text{Lie}(X_1, \dots, X_m, Y)(z) = \mathbb{R}^{N+1}$ for any $z \in \Omega$;
- $\mathcal{L}$ is a non totally degenerate operator, that is at least one of the vector fields $X_1(z), \dots, X_m(z)$ is non zero $\forall z \in \mathbb{R}^{N+1}$.

With these assumptions he built a family of bounded open regular sets by a general method that relies on a barrier argument.

Our results are a generalization of his work, taking into account totally degenerate differential operator and requiring as mild hypothesis the only existence of a basis of resolutive set

[3] see J.-M. Bony, Ann. Inst. Fourier, 1969

Perron's Method: resolutive sets for $\mathcal{L}$

We define now the family of resolutive sets we are going to consider.

Given $z_0 = (x_0, t_0) \in \mathbb{R}^{N+1}$, $T > t_0$ and $r > 0$ let

$$Q_{r,T}(z_0) := \left\{ z : t \in (t_0, T), |\delta_0(r^{-1/2})(E(-t)x, E(-t_0)x_0)| < 1 \right\},$$

and denote by $\partial_P Q_{r,T}(z_0)$ its parabolic boundary.

As proven by Montanari[4] that there exists a solution u to

$$\begin{cases} \mathcal{L}u = 0, \text{ in } Q_{r,T}(z_0), \\ u_{|\partial_P Q_{r,T}(z_0)} = \varphi. \end{cases}$$

[4]see A. Montanari, Bollettino U.M.I. 1996

Perron's Method

Relying on the Perron's Method[5] and assuming as basis of the Euclidean topology on $\mathbb{R}^{N+1}$ the family

$$\mathscr{C} := \left\{ Q_{r,T}(z_0) : z_0 \in \mathbb{R}^{N+1}, T > t_0, r > 0 \right\}.$$

we get

Theorem (Manfredini, P. & Polidoro)

Every open set $\Omega \subseteq \mathbb{R}^{N+1}$ is resolutive and $H_\varphi^\Omega \in C^\infty(\Omega)$ is a classical solution to $\mathcal{L}u = 0$.

[5] see C. Constantinescu & C. Cornea, Springer-Verlarg 1972

Boundary value problem

In general, the boundary datum it is not always attained by the generalized solution H_φ^Ω. Then, it is important to obtain equivalent or stronger conditions that ensure $\mathcal{L}$-regularity.

Our main results are the following

- Wiener's Criterion[6];
- A Zaremba-type cone condition[7].

[6] see A. Kogoj, E. Lanconelli & G. Tralli, Discrete Contin. Dyn. Syst. Ser. A, 2018

[7] see M. Manfredini, Adv. Differential Equations, 1997

Wiener's Criterion

We call reduit function of 1 over U the function

$$\mathbf{R}_U^1 := \inf\{v \text{ "hyperharmonic" in } \mathbb{R}^{N+1},\ v \geq 0,\ v \geq 1 \text{ in } U\},$$

and balayage of 1 over U

$$\widehat{\mathbf{R}}_U^1(z) := \liminf_{\xi \to z} \mathbf{R}_U^1(\xi).$$

Given $n \in \mathbb{N}$, $\lambda \in (0,1)$ and $z_0 \in \partial\Omega$ define

$$\Omega_n^c(z_0) := \left\{ z \in \Omega^c : \lambda^{-n \log n} \leq \Gamma(z; z_0) \leq \lambda^{-(n+1)\log(n+1)} \right\} \cup \{z_0\}.$$

Theorem (Manfredini, P. & Polidoro)

Let Ω be an open subset of $\mathbb{R}^{N+1}$ and let $z_0 \in \partial\Omega$. Then,

$$z_0 \text{ is } \mathcal{L}\text{-regular} \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} \widehat{\mathbf{R}}^1_{\Omega_n^c(z_0)}(z_0) = +\infty.$$

Lemma

Every singleton $\{z_0\}$, $z_0 \in \mathbb{R}^{N+1}$, is a polar set in $\mathbb{R}^{N+1}$, i.e. there exists a non-negative "hyperharmonic" function $p \neq \infty$ such that $p(z_0) = +\infty$.

Together with the Lemma above we apply a characterization of boundary $\mathcal{L}$-regularity proven in the abstract setting of Potential Theory, which generalizes the analogous result due to Negrini & Scornazzani[8] for *non-totally degenerate* Kolmogorov operators, getting

Lemma

Let $\Omega \subseteq \mathbb{R}^{N+1}$ and let $z_0 \in \partial\Omega$. Then,

$$z_0 \text{ is } \mathcal{L}\text{-regular} \quad \Leftrightarrow \quad \lim_{r \to 0^+} \widehat{\mathbf{R}}^1_{B_r(z_0) \smallsetminus \Omega}(z_0) > 0.$$

[8] P. Negrini & V. Scornazzani, Bollettino U.M.I. Analisi Funzionale e Applicazioni Serie VI 1984

The Zaremba-type cone condition

For $T > 0$ and compact $K \subset \mathbb{R}^N$ with positive Lebesgue measure we call *$\mathcal{L}$-cone of vertex $z_0 = (x_0, 0)$* the set

$$C_{z_0} := \left\{ (\delta_0(r)x + x_0, -r^2 T) : x \in K, 0 \leq r \leq 1 \right\}.$$

Given Ω, we say that there exists an exterior cone with vertex in $z_0 \in \partial\Omega \cap \{t = 0\}$ if there exists C_{z_0} such that $C_{z_0} \subseteq \Omega^c$.

Proposition (Manfredini, P. & Polidoro)

Let Ω be an open set of $\mathbb{R}^{N+1}$ and let $z_0 = (x_0, 0) \in \partial\Omega$. If there exists an exterior cone C_{z_0} with vertex at z_0, then z_0 is $\mathcal{L}$-regular.

The requirement that the time coordinate of z_0 is zero is needed because the exterior cone criterion proven by Manfredini still works for every point $z_0 = (x_0, t_0)$ with $t_0 \neq 0$.