# Poisson process and sharp constants in $L^p$-estimates for a class of degenerate Kolmogorov operators

Enrico Priola

*Pavia*

with L. Marino (IMPAN)
S. Menozzi (Evry)

INDAM Meeting "Kolmogorov operators and their Applications",
13-17 june 2022 Cortona

# Introduction

I will present a method based on the Poisson process that can be applied to study

i) the heat equation in $\mathbb{R}^n$ (Krylov-P.17)

ii) more general non-degenerate parabolic PDEs (and other PDEs) (Krylov-P.17)

iii) some degenerate parabolic PDEs (Marino-Menozzi-P. arxiv 21)

....

I concentrate on deriving $L_p$ (or Sobolev) estimates but also Schauder estimates and other estimates can be investigated

This can also be seen as a probabilistic approach to $L_p$-estimates starting from one dimensional estimates

# The one dimensional heat equation

$$\boxed{\partial_t u(t,x) = D_x^2 u(t,x) + f(t,x), \qquad u(0,\cdot) = 0} \tag{1}$$

for $t \in (0,T)$, $x \in \mathbb{R}$.
We consider the problem in the integral form:

$$u(t,x) = \int_0^t (D_x^2 u(s,x) + f(s,x))ds, \ \ t \in [0,T], \ x \in \mathbb{R}.$$

Fix $p \in (1,\infty)$. One knows (see for instance Ladyzhenskaya-Solonnikov-Uraltseva 1968):

if $f \in B_c((0,T), C_0^\infty(\mathbb{R}))$, then there is a unique solution $u(t,x)$ such that $u$ is bounded and continuous on $[0,T] \times \mathbb{R}$; $u(t,\cdot) \in C^\infty(\mathbb{R})$, for any $t \in [0,T]$, furthermore:

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}} |u(t,x)| \leqslant T \sup_{(t,x) \in (0,T) \times \mathbb{R}} |f(t,x)|, \tag{2}$$

$$\|D_x^2 u\|^p_{L_p((0,T)\times\mathbb{R})} \leqslant N_p \|f\|^p_{L_p((0,T)\times\mathbb{R})}, \tag{3}$$

where $L_p$-spaces are defined with respect to Lebesgue measure and $N_p$ are some constants.

**Notation** For a real-valued function $f(t,x)$, $t \in (0,T)$, $x \in \mathbb{R}^d$, write

$$f \in B_c((0,T), C_0^\infty(\mathbb{R}^d))$$

if $f$ is a Borel bounded function, such that

$$f(t,\cdot) \in C_0^\infty(\mathbb{R}^d)$$

for any $t \in (0,T)$. For any $n = 0,1,...$, the $C^n(\mathbb{R}^d)$-norms of $f(t,\cdot)$ are bounded on $(0,T)$, and the supports of $f(t,\cdot)$ belong to the same ball.

Moreover, we require in addition that, for any $x \in \mathbb{R}^d$, the mapping:

$$t \mapsto f(t,x)$$

is a *piece-wise continuous* function on $[0,T]$.

**Remark** Also Schauder estimates like

$$\sup_{t\in[0,T]} [D_x^2 u(t,\cdot)]_{C^\alpha(\mathbb{R})} \leqslant N_0(\alpha) \sup_{t\in(0,T)} [f(t,\cdot)]_{C^\alpha(\mathbb{R})},$$

can be considered

Using the previous one-dimentional result and the Poisson process one derives:

**Theorem (I Krylov-P. 17)**

*For any $f \in B_c((0,T), C_0^\infty(\mathbb{R}^d))$ there exists a unique continuous and bounded on $[0,T] \times \mathbb{R}^d$ solution $u(t,x)$ smooth in the x-variable of the equation:*

$$\partial_t u(t,x) = \triangle u(t,x) + f(t,x), \quad u(0,\cdot) = 0$$

*in $(0,T) \times \mathbb{R}^d$ such that, for any $t \in [0,T]$, and, for any $i,j = 1,...,d$ and unit vector $l \in \mathbb{R}^d$, we have:*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |u(t,x)| \leqslant T \sup_{(t,x)\in(0,T)\times\mathbb{R}^d} |f(t,x)| \quad \textit{(Max. Principle)},$$

$$\|D_l^2 u\|^p_{L_p((0,T)\times\mathbb{R}^d)} \leqslant N_p \|f\|^p_{L_p((0,T)\times\mathbb{R}^d)}, \tag{4}$$

*where $N_p$ are the previous one-dimensional constants.*

No analytic methods are available up to now to prove the previous result.

## II Theorem in [Krylov, P.17]

Let $a(t) = (a^{ij}(t))$ be a $d \times d$ symmetric matrix-valued continuous on $(0, T)$ (more generally, locally bounded Borel measurable) such that

$$a^{ij}(t)\lambda^i\lambda^j \geqslant |\lambda|^2, \quad t \in (0, T), \ \lambda \in \mathbb{R}^d.$$

For any $f \in B_c((0, T), C_0^\infty(\mathbb{R}^d))$ there exists a unique bounded and continuous in $[0, T] \times \mathbb{R}^d$ solution $u(t, x)$ of the equation (we use $D_{ij}u = \partial^2_{x_i x_j} u$):

$$\partial_t u(t, x) = a^{ij}(t) D_{ij} u(t, x) + f(t, x), \quad u(0, \cdot) = 0$$

in $(0, T) \times \mathbb{R}^d$ such that, for any $t \in [0, T]$, $u(t, \cdot) \in C^\infty(\mathbb{R}^d)$ and, for any $i, j = 1, ..., d$ and unit vector $l \in \mathbb{R}^d$, we have:

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |u(t, x)| \leqslant T \sup_{(t,x)\in(0,T)\times\mathbb{R}^d} |f(t, x)| \quad \text{(Max. Principle)},$$

$$\|D_l^2 u\|^p_{L_p((0,T)\times\mathbb{R}^d)} \leqslant N_p \|f\|^p_{L_p((0,T)\times\mathbb{R}^d)}, \tag{5}$$

where $N_p$ are the previous one-dimensional constants. □

**Remark.** Estimates are independent on the dimension and on the supremum norm of $a(t)$.

# Idea of proof of I Theorem

We are considering in $(0,T) \times \mathbb{R}^d$:

$$\partial_t v(t,x) = \Delta v(t,x) + f(t,x), \quad u(0,\cdot) = 0 \tag{6}$$

and show that parabolic estimates hold true with the same one-dimensional constants.

We use the **Poisson process** and random PDEs

Take a sequence $\tau_1 = \tau_1(\omega), \tau_2 = \tau_2(\omega), \ldots$ of independent random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ with common exponential distribution with parameter $\lambda > 0$, so that $P(\tau_n > t) = e^{-\lambda t}$ for $t \geqslant 0$ and $n = 1, 2.....$ Define

$$\sigma_0 = 0, \quad \sigma_n = \sum_{i=1}^{n} \tau_i, \quad n = 1, 2, ..., \quad \pi_t = \pi_t(\omega) = \sum_{n=1}^{\infty} I_{\sigma_n \leqslant t}$$

(where $I_{\sigma_n \leqslant t}$ denotes the indicator function of the event $\{\sigma_n \leqslant t\}$). We see that $\pi_t$ is the number of consecutive sums of $\tau_i$ which lie on $[0, t]$.

The counting process $\pi_t$ is known as a Poisson process with parameter $\lambda$.

# The Poisson process

For $0 \leqslant s \leqslant t < \infty$ and $k = 0, 1, \ldots$ it holds that

$$P(\pi_t - \pi_s = k) = \frac{[\lambda(t-s)]^k}{k!} e^{-\lambda(t-s)},$$

$\lambda > 0$ and moreover, for any $t > s \geqslant 0$, $\pi_t - \pi_s$ is independent of the $\sigma$-algebra generated by all $\pi_r$, when $r \in [0, s]$.

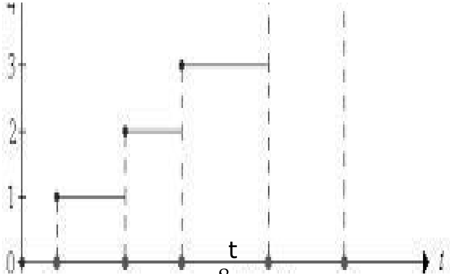
Let $\pi_{s-} = \lim_{t \uparrow s} \pi_t$, $s > 0$.

Let $h \in \mathbb{R}$. We do first some elementary computations on the generator of

$$h\pi_t$$

(below in the picture $h = 1$):

$$\mathcal{A}f(x) = \lambda(f(x+h) - f(x)), \quad x \in \mathbb{R}.$$



## Generator of $h\pi_t$ with parameter $\lambda > 0$

Let $u_0 \in C_b(\mathbb{R})$ be a bounded continuous function. Let $\omega \in \Omega$ be such that $n = \pi_t(\omega)$. We have, for $x \in \mathbb{R}$, $t > 0$, omitting $\omega$,

$$u_0(x + h\pi_t) - u_0(x)$$

$$= u_0(x + h\pi_{\sigma_n -} + h) - u_0(x + h\pi_{\sigma_n -}) + u_0(x + h\pi_{\sigma_{n-1} -} + h) - u_0(x + h\pi_{\sigma_{n-1} -})$$

$$... + u_0(x + h\pi_{\sigma_1 -} + h) - u_0(x + h\pi_{\sigma_1 -})$$

$$= \sum_{k=1}^{n} \big(u_0(x + h\pi_{\sigma_k -} + h) - u_0(x + h\pi_{\sigma_k -})\big)$$

$$= \sum_{\sigma_k \leqslant t} \int_0^t \big(u_0(x + h\pi_{s-} + h) - u_0(x + h\pi_{s-})\big)\delta_{\sigma_k}(ds)$$

$$= \int_0^t \big(u_0(x + h\pi_{s-} + h) - u_0(x + h\pi_{s-})\big)d\pi_s \quad \text{(Lebesgue-Stieltjes integral).}$$

Applying expectation:

$$E\Big[\int_0^t \big(u_0(x + h\pi_{s-} + h) - u_0(x + h\pi_{s-})\big)d\pi_s\Big] = \lambda \int_0^t E\big[u_0(x + h\pi_s + h) - u_0(x + h\pi_s)\big]ds$$

Set $v_t(x) = v(t, x) = E[u_t(x + h\pi_t)]$. Then

$$v_t(x) - u_0(x) = \lambda \int_0^t \big[v_s(x + h) - v_s(x)\big]\, ds,$$

i.e., $\partial_t v_t(x) = \lambda(v_t(x + h) - v_t(x))$ □

# The proof for $\partial_t v(t,x) = \Delta v(t,x) + f(t,x)$ in $(0,T) \times \mathbb{R}^d$

We consider $d = 2$. The general case comes from induction. Thus we need to pass from parabolic estimates with $d = 1$ to estimates with $d = 2$.

**I step.** Take $f(t,x,y)$ in $B_c((0,T), C_0^\infty(\mathbb{R}^2))$ and for each $\omega \in \Omega$ and $y \in \mathbb{R}$ solve:

$$\partial_t u(t,x,y,\omega) = D_x^2 u(t,x,y,\omega) + f(t,x,y - h\pi_t(\omega)) \tag{7}$$

with zero initial data, where $h \in \mathbb{R}$ is a parameter. [ We often do not indicate the dependence on $\omega$. Moreover, we also drop the dependence on $h$.]

There exists a unique solution $u(t,x,y)$, depending on $y$, $h$ and $\omega$ as parameters, such that main parabolic estimates hold for each $\omega$, $h$ and $y \in \mathbb{R}$ *with the same constants* if we replace $u(t,x)$ and $f(t,x)$ with $u(t,x,y)$ and $f(t,x,y - h\pi_t)$, respectively.

Furthermore, since $f \in B_c((0,T), C_0^\infty(\mathbb{R}^2))$, one can prove that $u(t,x,y)$ *is uniformly continuous with respect to y uniformly with respect to* $\omega, t$, *h, and x*

Let us see what equation is verified by

$$u(t,x,y + h\pi_t)$$

By considering $u(t,x,y+h\pi_t)$ on each interval $[\sigma_n, \sigma_{n+1})$ on which $h\pi_t$ is constant, one easily derives that

$$u(t,x,y+h\pi_t) = \int_0^t [D_x^2 u(s,x,y+h\pi_s) + f(s,x,y)]\, ds + \int_{(0,t]} g(s,x,y)\, d\pi_s \tag{8}$$

$$= \int_0^t [D_x^2 u(s,x,y+h\pi_s) + f(s,x,y)]\, ds + \sum_{\sigma_n \leqslant t} g(\sigma_n, x, y),$$

where

$$g(s,x,y) = u(s,x,y+h+h\pi_{s-}) - u(s,x,y+h\pi_{s-}) \tag{9}$$

is the jump of $u(t,x,y+h\pi_t)$ as a function of $t$ at moment $s$ if $\pi_t$ has a jump at $s$.

Recall $\pi_{s-} = \lim_{t\uparrow s} \pi_t$, $s > 0$.
For instance, if $t \in [\sigma_1, \sigma_2)$ we have

$$u(t,x,y+h\pi_t) = u(t,x,y+h) = \int_{\sigma_1}^t [D_x^2 u(s,x,y+h) + f(s,x,y)]\, ds$$

$$+u(\sigma_1,x,y+h) - u(\sigma_1,x,y) + \int_0^{\sigma_1} [D_x^2 u(s,x,y) + f(s,x,y)]\, ds.$$

Let

$$v(t,x,y) := E[u(t,x,y+h\pi_t)].$$

We get, for any $t \in (0,T)$, $x, y \in \mathbb{R}$,

$$v(t,x,y) = \int_0^t \big(D_x^2 v(s,x,y) + \lambda[v(s,x,y+h) - v(s,x,y)] + f(s,x,y)\big) ds.$$

**II step.** *Let $f \in B_c((0,T), C_0^\infty(\mathbb{R}^2))$, $h \in \mathbb{R}$ and $\lambda > 0$. Then there exists a unique bounded continuous function $v(t,x,y)$, $t \in [0,T]$, $x, y \in \mathbb{R}$, satisfying*

$$\partial_t v(t,x,y) = D_x^2 v(t,x,y) + \lambda[v(t,x,y+h) - v(t,x,y)] + f(t,x,y) \tag{10}$$

*for $t \in (0,T)$, $x, y \in \mathbb{R}$, with zero initial condition and such that $v(t,\cdot,y) \in C^\infty(\mathbb{R})$ for any $t \in (0,T)$, $y \in \mathbb{R}$.*

*Furthermore with $N_p$ as in the one-dimensional estimates :*

$$\sup_{(t,z)\in[0,T]\times\mathbb{R}^2} |v(t,z)| \leqslant T \sup_{(t,z)\in(0,T)\times\mathbb{R}^2} |f(t,z)|,$$

$$\|D_x^2 v\|^p_{L_p((0,T)\times\mathbb{R}^2)} \leqslant N_p \|f\|^p_{L_p((0,T)\times\mathbb{R}^2)}$$

Recall that by uniqueness $v(t,x,y) := E[u(t,x,y+h\pi_t)]$.

Let us only check

$$\|D_x^2 v\|^p_{L_p((0,T)\times\mathbb{R}^2)} \leqslant N_p \|f\|^p_{L_p((0,T)\times\mathbb{R}^2)}.$$

We compute, using also Jensen inequality and the Fubini theorem,

$$\begin{aligned}
\|D_x^2 v\|^p_{L_p} &= \int_{[0,T]\times\mathbb{R}^2} \Big|E\Big[D_x^2 u(t,x,y+h\pi_t)\Big]|^p dtdxdy \\
&\leqslant \int_0^T dt \int_{\mathbb{R}^2} E\Big[|D_x^2 u(t,x,y+h\pi_t)|^p\Big] dxdy \\
&= \int_0^T dt \int_{\mathbb{R}^2} E\Big[|D_x^2 u(t,x,z|^p\Big] dxdz
\end{aligned}$$

$$= \int_{\mathbb{R}} dz \int_0^T dt \int_{\mathbb{R}} E\Big[|D_x^2 u(t,x,z|^p\Big] dx \leqslant N_p E \int_{\mathbb{R}} dx \int_0^T dt \int_{\mathbb{R}} |f(t,x,z-h\pi_t)|^p dz$$

$$\leqslant N_p \int_{\mathbb{R}} dz \int_0^T dt \int_{\mathbb{R}} |f(t,x,z|^p dx.$$

We have also used invariance by translation of the Lebesgue measure.

**III step.** By repeating the above argument, we consider

$$\partial_t \tilde{v}(t,x,y) = D_x^2 \tilde{v}(t,x,y) + \lambda[\tilde{v}(t,x,y+h) - \tilde{v}(t,x,y)] + f(t,x,y+h\pi_t))$$

so that

$$w(t,x,y) := E[\tilde{v}(t,x,y-h\pi_t)]$$

satisfies

$$\partial_t w(t,x,y) = D_x^2 w(t,x,y)$$
$$+ \lambda[w(t,x,y+h) - 2w(t,x,y) + w(t,x,y-h)] + f(t,x,y) \tag{11}$$

and admits the same parabolic estimates as before (with the same constants)

Then we take $\lambda = h^{-2}$ in (11) and let $h \downarrow 0$.

*By Ascoli-Arzela, solutions* $w = w_h$ *of* (11) *with* $\lambda = h^{-2}$ *converge to* $v(t,x,y)$*, that is infinitely differentiable with respect to* $(x,y)$ *for any* $t$ *with any derivative continuous and bounded on* $[0,T] \times \mathbb{R}^2$*, (equals zero for* $t = 0$*);* it satisfies

$$\partial_t v(t,x,y) = \Delta_{xy} v(t,x,y) + f(t,x,y) = D_{xx}^2 v(t,x,y) + D_{yy}^2 v(t,x,y) + f(t,x,y) \tag{12}$$

and all the parabolic estimates hold true for such $v$ with the same constants.

Bounded continuous in $[0,T] \times \mathbb{R}^2$ solutions of (12) having continuous second-order derivatives with respect to $(x,y)$ and vanishing at $t = 0$ are unique, and we get that, for any such solution the previous parabolic estimates hold true with the same constants.

**IV step.** Take a unit vector $l_1 \in \mathbb{R}^2$ and a unit vector $l_2 \in \mathbb{R}^2$ orthogonal to $l_1$. Let $S$ be an orthogonal transformation of $\mathbb{R}^2$ such that $Se_i = l_i$, $i = 1, 2$, where $e_1, e_2$ is the standard basis in $\mathbb{R}^2$, and set $f(t, xe_1 + ye_2) = f(t, x, y)$, $v(t, xe_1 + ye_2) = v(t, x, y)$,

$$S(x, y) = xl_1 + yl_2, \quad g(t, x, y) = f(t, S(x, y)), \quad w(t, x, y) = v(t, S(x, y)).$$

Since the Laplacian is rotation invariant, we have

$$\partial_t w(t, x, y) = \Delta w(t, x, y) + g(t, x, y)$$

and, since $g$ is as regular as $f$ and

$$D_x^2 w(t, x, y) = (D_{l_1}^2 v)(t, S(x, y)) = (D_{l_1}^2 v)(t, xl_1 + yl_2),$$

where $D_l^2 = l^i l^j D_{ij}$ and $D_i = \partial / \partial x^i, \quad D_{ij} = D_i D_j$.
Since the Jacobian of the above $S(x, y)$ equals one, for any unit vector $l \in \mathbb{R}^2$

$$\int_0^T \int_{\mathbb{R}^2} |D_l^2 v(t, z)|^p \, dzdt \leqslant N_p \int_0^T \int_{\mathbb{R}^2} |f(t, z)|^p \, dzdt. \quad \square$$

# On the proof for $\partial_t v(t,x) = Tr(a(t)D_x^2 v(t,x)) + f(t,x)$

$$\partial_t v(t,x) = Tr(a(t)D_x^2 v(t,x)) + f(t,x), \quad v(0,\cdot) = 0.$$

To explain the main idea:

$$\partial_t v(t,x) = \triangle v(t,x) + Tr(c(t)D_x^2 v(t,x)) + f(t,x), \quad \text{with } c(t) = a(t) - I.$$

$$\partial_t u(t,x) = \triangle u(t,x) + f(t,x), \qquad t \in (0,T),\ x \in \mathbb{R}^d. \tag{13}$$

Let $h \in \mathbb{R}$ and consider the unit vector $e_1 \in \mathbb{R}^d$. We define

$$b_t = \int_{(0,t]} \sqrt{c(r)}\, e_1 d\pi_r = \sum_{\sigma_k \leqslant t,\ k \geqslant 1} \sqrt{c(\sigma_k)}\, e_1.$$

If we replace $f(t,x)$ with $f(t,x - hb_t)$, for each $\omega$, in eq. (13), one derives that $u(t,x+hb_t)$ satisfies

$$u(t,x+hb_t) = \int_0^t [\triangle u(s,x+hb_s) + f(s,x)]\, ds + \int_{(0,t]} g(s,x)\, d\pi_s,$$

where

$$g(s,x) := u(s,x+h\sqrt{c(s)}\, e_1 + hb_{s-}) - u(s,x+hb_{s-}).$$

Let $v(t,x) = Eu(t, x + hb_t)$.

We arrive at

$$\partial_t v(t,x) = \triangle v(t,x) + \lambda[v(t, x + h\sqrt{c(t)}\, e_1) - v(t,x)] + f(t,x).$$

After that we solve

$$\partial_t w(t,x) = \triangle w(t,x) + \lambda[w(t, x + h\sqrt{c(t)}\, e_1) - w(t,x)] + f(t, x + hb_t)$$

and repeating the previous arguments we conclude that for each $h > 0$ there exists a unique solution $u_h(t,x)$ on $[0,T] \times \mathbb{R}^d$ to

$$\partial_t u_h(t,x) = \triangle u_h(t,x) + f(t,x)$$
$$+ h^{-2}[u_h(t, x + h\sqrt{c(t)}\, e_1) - 2u_h(t,x) + u_h(t, x - h\sqrt{c(t)}\, e_1)]$$

in $(0,T) \times \mathbb{R}^d$ with zero initial condition and for which all parabolic estimates hold true. Passing to the limit, as $h \to 0^+$, we get

$$\partial_t w(t,x) = \triangle w(t,x) + \langle D^2 w(t,x)\sqrt{c(t)}\, e_1, \sqrt{c(t)}\, e_1\rangle + f(t,x).$$

By adding other terms we find

$$\partial_t w(t,x) = \triangle w(t,x) + \sum_{k=1}^{d} \langle D^2 w(t,x)\sqrt{c(t)}\, e_k, \sqrt{c(t)}\, e_k\rangle + f(t,x). \quad \square$$

## The Kolmogorov example (a degenerate parabolic equation)

Let us consider the Kolmogorov example on $\mathbb{R}^2$ (cf. [Kolmogorov 34]) in the following form:

$$\begin{cases} \partial_t u(t,x,y) = \partial^2_{xx} u(t,x,y) + x\partial_y u(t,x,y) + f(t,x,y), \text{ on } [0,T] \times \mathbb{R}^2; \\ \\ u(0,x,y) = 0, \text{ on } \mathbb{R}^2; \end{cases} \tag{14}$$

where $(x,y)$ in $\mathbb{R}^2$, $T > 0$ is fixed and $f \in C_0^\infty((0,T) \times \mathbb{R}^2)$.

We have existence of a unique classical bounded solution and according to [Bramanti-Cerutti-Manfredini 96] there exists $C_p > 0$ independent of $u$ and $f$ such that

$$\|\partial^2_{xx} u\|_{L^p((0,T)\times\mathbb{R}^2)} \leqslant C_p \, \|f\|_{L^p((0,T)\times\mathbb{R}^2)} = C_p \|\partial_t u - L^{\mathrm{Kol}} u\|_{L^p((0,T)\times\mathbb{R}^2)}. \tag{15}$$

where $p \in (1, +\infty)$.

# A more general degenerate equation (Marino-Menozzi-P.)

We consider a perturbation like

$$\begin{cases} \partial_t w(t,x,y) \;=\; \partial^2_{xx} w(t,x,y) + x\partial_y w(t,x,y) + s(t)\partial^2_{yy} w(t,x,y) + f(t,x,y), \\ \\ w(0,x,y) \;=\; 0, \text{ on } \mathbb{R}^2; \end{cases}$$

where $s(t)$ is a continuous and non-negative function defined on $[0,T]$

We prove that the unique bounded solution $w$ verifies

$$\|\partial^2_{xx} w\|_{L^p((0,T)\times\mathbb{R}^2)} \leqslant C_p \; \|f\|_{L^p((0,T)\times\mathbb{R}^2)}$$

with the same constant $C_p$ as in (15) (hence, independently of $s(t)$).

# Comments

- We use the previous probabilistic approach based on the Poisson process

- The result has two parts:

(i) show that there exists $M_p > 0$ such that the solution $w$ to the perturbed equation verifies:

$$\|\partial^2_{xx} w\|_{L^p((0,T)\times\mathbb{R}^2)} \leqslant M_p \, \|f\|_{L^p((0,T)\times\mathbb{R}^2)}$$

(ii) show that infact $M_p = C_p$

- Actually, we do not know analytic methods to get even (i).

It remains a challenging open problem to have a purely analytic proof of our regularity results.

# Related Schauder estimates

Starting from anisotropic Schauder estimates like

$$\sup_{0 \leqslant t \leqslant T} \|u(t,\cdot)\|_{C^{2+\beta}_{b,d}} \leqslant C_\beta \sup_{0 \leqslant t \leqslant T} \|f(t,\cdot)\|_{C^{\beta}_{b,d}}. \tag{16}$$

for $\beta \in (0,1)$ [cf. Lunardi 96] we can derive

$$\sup_{0 \leqslant t \leqslant T} \|w(t,\cdot)\|_{C^{2+\beta}_{b,d}} \leqslant C_\beta \sup_{0 \leqslant t \leqslant T} \|f(t,\cdot)\|_{C^{\beta}_{b,d}}.$$

with the same constant $C_\beta$ as before.

Note that (see also [Lanconelli-Polidoro94] and [Pascucci02]).

$$C^{\gamma}_{b,d} = C^{\gamma}_{b,d}(\mathbb{R}^2),$$

$\gamma = 2 + \beta$ or $\gamma = \beta$ is a space of bounded and Hölder continuous functions with respect to the distance:

$$d(z,z') := |x - x'| + |y - y'|^{1/3}, \qquad z = (x,y),\ z' = (x',y') \in \mathbb{R}^2 \quad \square$$

**Remark.** We can pass from other estimates available for $u$ to corresponding estimates for $w$ mantaining the same constant.

However in the sequel I will concentrate on the initial $L^p$-estimates

# A matrix notation for the initial Kolmogorov equation

For the sequel: writing $z = (x, y)$, the initial equation can be rewritten as:

$$\partial_t u(t,z) \; = \; \langle Az, Du(t,z)\rangle + \mathrm{tr}\left(BD^2 u(t,z)\right) + f(t,z), \quad u(0,z) = 0$$

for matrixes $A, B$ in $\mathbb{R}^2 \otimes \mathbb{R}^2$ given by

$$A \; := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad B \; := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \; \textit{and} \qquad D_z^2 = D^2 = \begin{pmatrix} \partial^2_{xx} & \partial^2_{xy} \\ \partial^2_{xy} & \partial^2_{yy} \end{pmatrix}$$

Here $D = D_z$ is the gradient. Note that

$$\|B(D^2 u)\, B\|_{L^p((0,T)\times\mathbb{R}^2)} \; \leqslant \; C_p \|f\|_{L^p((0,T)\times\mathbb{R}^2)},$$

is equivalent to the initial estimate

$$\|\partial^2_{xx} u\|_{L^p((0,T)\times\mathbb{R}^2)} \leqslant C_p \; \|f\|_{L^p((0,T)\times\mathbb{R}^2)}$$

## $L^p$-estimates for degenerate Ornstein-Uhlenbeck operators

Let us now describe the more general framework we are going to consider .

$\mathbb{R}^N = \mathbb{R}^{d_0} \times \mathbb{R}^{d_1}$, $d_0, d_1$ *non-negative* integers such that $d_0 + d_1 = N$, $d_0 \geqslant 1$.

Let us introduce the non-negative, symmetric matrix $B$ in $\mathbb{R}^N \otimes \mathbb{R}^N$ given by

$$B = \begin{pmatrix} B_0 & 0 \\ 0 & 0 \end{pmatrix},$$

where $B_0$ is a symmetric, positive definite matrix in $\mathbb{R}^{d_0} \otimes \mathbb{R}^{d_0}$ such that

$$\nu \sum_{i=1}^{d_0} \xi_i^2 \leqslant \sum_{i,j=1}^{d_0} (B_0)_{ij} \xi_i \xi_j \leqslant \frac{1}{\nu} \sum_{i=1}^{d_0} \xi_i^2,$$

for all $\xi \in \mathbb{R}^{d_0}$, for some $\nu > 0$. We will consider possibly degenerate Ornstein-Uhlenbeck operators which generalizes $L^{Kol}$:

$$L^{\mathrm{ou}} f(z) = \mathrm{Tr}(BD^2 f(z)) + \langle Az, Df(z) \rangle, \quad z = (x, y) \in \mathbb{R}^{d_0 + d_1} = \mathbb{R}^N, \tag{17}$$

for $A$ in $\mathbb{R}^N \otimes \mathbb{R}^N$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^N$.

# The Kalman condition for OU

We assume the Kalman condition:

**[K]** There exists a non-negative integer $k$, such that

$$\mathrm{Rank}[B, AB, \ldots, A^k B] = N, \tag{18}$$

where $[B, AB, ..., A^k B]$ is $\mathbb{R}^N \otimes \mathbb{R}^{N(k+1)}$ whose blocks are $B, AB, \ldots, A^k B$. □

From the non-degeneracy of $B_0$, the above condition amounts to say that the vectors

$$\{e_1, \ldots, e_{d_0}, Ae_1, \ldots, Ae_{d_0}, \ldots, A^k e_1, \ldots, A^k e_{d_0}\} \quad \text{generate} \quad \mathbb{R}^N, \tag{19}$$

where $\{e_i\}_{i \in \{1, \cdots, d_0\}}$ are the first $d_0$ vectors of the canonical basis for $\mathbb{R}^N$.

Assumption **[K]** is equivalent to the Hörmander condition on the commutators ([Hormander 67]) ensuring the hypoellipticity of $\partial_t - L^{\mathrm{ou}}$.

Note that $L^{Kol}$ verifies **[K]**.

## Known $L^p$-estimates for OU (under the Kalman condition)

First one proves existence and uniqueness of bounded regular solutions to

$$\begin{cases} \partial_t u(t,z) = L^{\text{ou}} u(t,z) + f(t,z), & \text{on } (0,T)\times\mathbb{R}^N; \\ u(0,z) = 0, & \text{on } \mathbb{R}^N, \end{cases} \tag{20}$$

when $f$ belongs to $B_b\left(0,T;C_0^\infty(\mathbb{R}^N)\right)$, which contains $C_0^\infty((0,T)\times\mathbb{R}^N)$.

*Equation* (20) *will be understood in integral form as in the beginning.*

By [Bramanti-Cupini-Lanconelli-P 10] for any fixed $p$ in $(1,+\infty)$, there exists $C_p = C_p(\nu, A, d_0, d_1, T)$ such that

$$\|D_x^2 u\|_{L^p((0,T)\times\mathbb{R}^N)} \leqslant C_p \|\partial_t u - L^{\text{ou}} u\|_{L^p((0,T)\times\mathbb{R}^N)} = C_p \|f\|_{L^p((0,T)\times\mathbb{R}^N)}, \tag{21}$$

$D_x^2 u(t,z)$, $z$ in $\mathbb{R}^N$, $t\in[0,T]$, is the Hessian matrix in $\mathbb{R}^{d_0}\otimes\mathbb{R}^{d_0}$ with respect to the variable $x$.

Setting

$$B_I = \begin{pmatrix} I_{d_0,d_0} & 0_{d_0,d_1} \\ 0_{d_1,d_0} & 0_{d_1,d_1} \end{pmatrix}$$

the **known** $L^p$**-estimate** (21) can be rewritten in the following, equivalent way:

$$\begin{aligned} \|B_I D^2 u\, B_I\|_{L^p((0,T)\times\mathbb{R}^N)} &= \|D_x^2 u\,\|_{L^p((0,T)\times\mathbb{R}^N)} \\ \leqslant C_p\|\partial_t u - L^{\text{ou}} u\|_{L^p((0,T)\times\mathbb{R}^N)} &= C_p\|f\|_{L^p((0,T)\times\mathbb{R}^N)}, \end{aligned} \tag{22}$$

where $D^2 u = D_z^2 u$ represents the full Hessian matrix in $\mathbb{R}^N \otimes \mathbb{R}^N$ with respect to $z$.

# The main $L^p$-result

Fix a continuous map: $t \mapsto S(t) \in \mathbb{R}^N \otimes \mathbb{R}^N$ such that $S(t)$ is a **symmetric** and **non-negative definite**, $t \in [0,T]$; consider the following perturbation of $L^{\mathrm{ou}}$ :

$$\begin{aligned} L_t^{\mathrm{ou},S} f(z) &:= \mathrm{Tr}(BD^2 f(z)) + \mathrm{Tr}(S(t)D^2 f(z)) + \langle Az, Df(z)\rangle \\ &= L^{\mathrm{ou}} f(z) + \mathrm{Tr}(S(t)D^2 f(z)), \end{aligned} \tag{23}$$

$z = (x,y)$ is in $\mathbb{R}^{d_0+d_1} = \mathbb{R}^N$. Let $u_S$ be the solution of the Cauchy problem

$$\begin{cases} \partial_t u_S(t,z) = L_t^{\mathrm{ou},S} u_S(t,z) + f(t,z), & \text{on } (0,T) \times \mathbb{R}^N; \\ u_S(0,z) = 0, & \text{on } \mathbb{R}^N, \end{cases} \tag{24}$$

### Theorem (Marino-Menozzi-P.21)

*Let us consider* (24) *with* $f \in B_b\left(0,T; C_0^\infty(\mathbb{R}^N)\right)$. *Then, there exists a unique solution* $u_S$ *of Cauchy Problem* (24) *which verifies, with the same constant* $C_p$ *as in* (22),

$$\begin{aligned} \|D_x^2 u_S\|_{L^p((0,T)\times\mathbb{R}^N)} &= \|B_I D^2 u_S B_I\|_{L^p((0,T)\times\mathbb{R}^N)} \\ \leqslant C_p \, \|\partial_t u_S - L_t^{\mathrm{ou},S} u_S\|_{L^p((0,T)\times\mathbb{R}^N)} &= C_p \, \|f\|_{L^p((0,T)\times\mathbb{R}^N)}. \end{aligned} \tag{25}$$

Possible applications to the study of the martingale problem.

# Some ideas from the proof

Fix a bounded solution $u$ to Cauchy Problem (20) **without** $S(t)$ and introduce

$$v(t,z) := u(t, e^{-tA}z), \quad u(t,z) = v(t, e^{tA}z).$$

This is a known transformation (see [Da Prato-Lunardi 95]; it allows to get rid of the drift term in the PDE satisfied by $v$:

$$\begin{aligned} f(t,z) &= \partial_t u(t,z) - L^{\mathrm{ou}} u(t,z) \\ &= v_t(t,e^{tA}z) + \langle Dv(t,e^{tA}z), Ae^{tA}z\rangle - \mathrm{Tr}\big(e^{tA}Be^{tA^*}D^2 v(t,e^{tA}z)\big) \\ &\quad - \langle Dv(t,e^{tA}z), Ae^{tA}z\rangle \\ &= v_t(t,e^{tA}z) - \mathrm{Tr}\big(e^{tA}Be^{tA^*}D^2 v(t,e^{tA}z)\big), \quad (t,z) \in (0,T) \times \mathbb{R}^N. \end{aligned}$$

Let $\tilde{f}(t,z) := f(t, e^{-tA}z)$, it follows that $v$ satisfies the PDE **(without drift)**:

$$\begin{cases} \partial_t v(t,z) = \mathrm{Tr}\big(e^{tA}Be^{tA^*}D^2 v(t,z)\big) + \tilde{f}(t,z) & \text{on } (0,T) \times \mathbb{R}^N; \\ v(0,z) = 0 & \text{on } \mathbb{R}^N. \end{cases} \tag{26}$$

In terms of the function $v$, the **known estimate** in (22) rewrites as:

$$\|B_I e^{tA^*} D^2 v(t, e^{tA}\cdot)\, e^{tA} B_I\|_{L^p((0,T)\times\mathbb{R}^N)} \leqslant C_p \|\tilde{f}(t, e^{tA}\cdot)\|_{L^p((0,T)\times\mathbb{R}^N)}, \tag{27}$$

using the notation $\|B_I e^{-tA^*} D^2 v(t, e^{tA}\cdot)\, e^{tA} B_I\|_{L^p((0,T)\times\mathbb{R}^N)}$ to stress the dependence on $t$ instead of the more precise

$$\|B_I e^{\cdot A^*} D^2 v(\cdot, e^{\cdot A}\cdot)\, e^{\cdot A} B_I\|_{L^p((0,T)\times\mathbb{R}^N)}.$$

By changing variable in the integrals, control (27) is equivalent to

$$\|B_I e^{tA^*} D^2 v(t, \cdot)\, e^{tA} B_I\|_{L^p((0,T)\times\mathbb{R}^N, m)} \leqslant C_p \|\tilde{f}\|_{L^p((0,T)\times\mathbb{R}^N, m)} \tag{28}$$

where $L^p((0,T)\times\mathbb{R}^N, m)$ denotes the $L^p$-norms w.r.t. the measure

$$m(dt, dx) = \det(e^{-At}) dt dx.$$

# The crucial step

Consider now the following more general Cauchy problem on $[0,T] \times \mathbb{R}^N$ involving $S(t)$:

$$\begin{cases} \partial_t w(t,z) = \mathrm{Tr}\big(e^{tA} B e^{tA^*} D^2 w(t,z)\big) + \mathrm{Tr}\big(e^{tA} S(t) e^{tA^*} D^2 w(t,z)\big) + \tilde{f}(t,z); \\ w(0,z) = 0, \end{cases} \tag{29}$$

Again one can prove existence and uniqueness of a bounded regular solution $w$.

Now the crucial step consists in adapting some arguments from [Krylov-P. 17] to derive that the same $L^p$-estimates in (28) still hold for $w$.

Precisely, we can prove that

$$\|B_I e^{tA^*} D^2 w(t,\cdot)\, e^{tA} B_I\|_{L^p((0,T)\times\mathbb{R}^N,m)} \leqslant \mathbf{C_p} \|\tilde{f}(t,\cdot)\|_{L^p((0,T)\times\mathbb{R}^N,m)}, \tag{30}$$

with the *same* constant $C_p$ appearing in (28) and so independently from the non-negative definite, symmetric matrices $S(t)$.

The last step is easy: it consists in coming back to the Ornstein-Uhlenbeck operators framework.

Namely, we introduce $\tilde{u}(t,z) := w(t,e^{tA}z)$ which solves

$$\begin{cases} \partial_t \tilde{u}(t,z) = L_t^{\mathrm{ou},S} \tilde{u}(t,z) + f(t,z), \ (t,z) \in (0,T) \times \mathbb{R}^N, \\ \tilde{u}(0,z) = 0, \ z \in \mathbb{R}^N. \end{cases}$$

Thus

$$\tilde{u} = u_S.$$

Noticing that $D^2 w(t,\cdot) = D^2[\tilde{u}(t,e^{-tA}\,\cdot)] = e^{-tA^*} D^2 \tilde{u}(t,e^{-tA}\cdot) e^{-tA}$ we thus get from (30) that

$$\|B_I D^2 \tilde{u}\, B_I\|_{L^p((0,T)\times\mathbb{R}^N)} \leqslant C_p \|f\|_{L^p((0,T)\times\mathbb{R}^N)}. \quad \square \tag{31}$$

## On the crucial step

Let $p \in (1,\infty)$ be fixed. Assume that there exists $R(t) \in \mathbb{R}^N \otimes \mathbb{R}^N$ depending continuously on $t \geqslant 0$ and a constant $C_p > 0$, such that for any $f$ in $B_b\left(0,T;C_0^\infty(\mathbb{R}^N)\right)$, the unique bounded solution

$$v = PDE(Q,f)$$

to

$$\begin{cases} \partial_t v(t,z) \;=\; \mathrm{Tr}\left(Q(t)D^2 v(t,z)\right) + f(t,z) & \text{on } (0,T)\times\mathbb{R}^N; \\ v(0,z) \;=\; 0 & \text{on } \mathbb{R}^N, \end{cases} \tag{32}$$

satisfies

$$\|R(t)^* D^2 v\, R(t)\,\|_{L^p((0,T)\times\mathbb{R}^N,\mathfrak{m})} \leqslant C_p \|f\|_{L^p((0,T)\times\mathbb{R}^N,\mathfrak{m})} \tag{33}$$

for some absolutely continuous measure $\mathfrak{m}$ w.r.t. the Lebesgue measure on $[0,T]\times\mathbb{R}^N$ such that $\mathfrak{m}(dt,dx) = g(t)dtdx$ for some Borel bounded function $g$

(note that in (28) we have $R(t) = e^{tA}B_I$, $\mathfrak{m}(dt,dx) = g(t)dtdx = \det(e^{-At})dtdx$).

We would like that a control like (33) also holds for the solution $w$ to the Cauchy Problem:

$$\begin{cases} \partial_t w(t,z) = \text{tr}\left(Q(t)D^2 w(t,z)\right) + \text{tr}\left(Q'(t)D^2 w(t,z)\right) + f(t,z), \text{ on } (0,T)\times\mathbb{R}^N; \\ w(0,z) = 0, \text{ on } \mathbb{R}^N, \end{cases} \tag{34}$$

Namely we have to prove the following result.

## Lemma (MMP21)

*Let us consider equations* (32) *and* (34) *where* $Q(t)$, $Q'(t)$ *are two continuous in time, non-negative definite matrices in* $\mathbb{R}^N \otimes \mathbb{R}^N$ *and* $f \in B_b\left(0,T; C_0^\infty(\mathbb{R}^N)\right)$. *Assume that estimate* (33) *holds as explained above. Then the solution* $w$ *to* (34) *verifies*

$$\|R(t)^* D^2 w\, R(t)\,\|_{L^p((0,T)\times\mathbb{R}^N,\mathfrak{m})} \leqslant C_p \|f\|_{L^p((0,T)\times\mathbb{R}^N,\mathfrak{m})}, \tag{35}$$

$p \in (1,\infty)$ *with the same constant* $C_p$ *as in* (33).

From this theorem, using the previous arguments we can derive the main regularity result for OU. □

## Final comments

We recall a maximal $L^p$-regularity result. For any fixed $p$ in $(1, +\infty)$, there exists $\tilde{C}_p > 0$ such that for any $f$ in $C_0^\infty((0,T) \times \mathbb{R}^{2d})$ the unique classical bounded solution $u$ of the Cauchy Problem verifies

$$\begin{aligned} \|(\Delta_y)^{\frac{1}{3}} u\|_{L^p((0,T)\times\mathbb{R}^{2d})} &\leqslant \tilde{C}_p \|f\|_{L^p((0,T)\times\mathbb{R}^{2d})} \\ &= \tilde{C}_p \|\partial_t u - L^{\mathrm{K}} u\|_{L^p((0,T)\times\mathbb{R}^{2d})}, \end{aligned} \tag{36}$$

where $(\Delta_y)^{\frac{1}{3}}$ denotes the fractional Laplacian with respect to the degenerate variables $y$ in $\mathbb{R}^d$.

(estimates holds for more general operators, see [ChenZhang19] and [HuangMenozziP19])

It turns out that this estimate is also stable under the previously described second order perturbation. Namely, for $u_S$ solving the perturbed equation we have

$$\begin{aligned} \|(\Delta_y)^{\frac{1}{3}} u_S\|_{L^p((0,T)\times\mathbb{R}^{2d})} &\leqslant \tilde{C}_p \|f\|_{L^p((0,T)\times\mathbb{R}^{2d})} \\ &= \tilde{C}_p \|\partial_t u_S - L^{\mathrm{K},S} u_S\|_{L^p((0,T)\times\mathbb{R}^{2d})}, \end{aligned} \tag{37}$$

where, again, $\tilde{C}_p$ is as before.