On the weak regularity theory for solutions to degenerate Kolmogorov equations

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Joint project with

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Kolmogorov Operators and their Applications

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Contents

Kolmogorov type operators

Aim

We want to study the regularity of weak solutions to

$$\boxed{\mathscr{L}u := \mathrm{div}(A(z)Du(z)) + Yu(z) + \langle b, Du(z)\rangle = f(z)}$$

where

- $z = (x,t) = (x_1, \ldots, x_N, t) \in \mathbb{R}^{N+1}$ and $1 \leq m_0 \leq N$,

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- $A(x,t) = (a_{ij}(x,t))_{1\leq i,j\leq N}$, with $a_{ij} = 0$ if $i > m_0$ or $j > m_0$,

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- $b(x,t) = (b_1(x,t), \dots, b_{m_0}(x,t), 0, \dots, 0)$.

Kolmogorov type operators

Assumptions

(H1) The matrix A_0 is symmetric with real measurable entries. Moreover, there exist two positive constants λ and Λ such that

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{m_0} a_{ij}(x,t)\xi_i\xi_j \leq \Lambda|\xi|^2$$

for every $(x,t) \in \mathbb{R}^{N+1}$ and $\xi \in \mathbb{R}^{m_0}$. The matrix B has constant entries.

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(H2) The *principal part operator* $\mathscr{K}$ *of* $\mathscr{L}$ is hypoelliptic and homogeneous of degree 2 with respect to the family of dilations $(\delta_r)_{r>0}$, where $\mathscr{K}$ is

$$\boxed{\mathscr{K}u(x,t) := \sum_{i=1}^{m_0} \partial^2_{x_i}u(x,t) + \sum_{i,j=1}^{N} b_{ij}x_j\partial_{x_i}u(x,t) - \partial_t u(x,t).}$$

Kolmogorov type operators

Assumptions

(H3) $f \in L^q(\Omega)$ and $b \in (L^q(\Omega))^{m_0}$ for some $q > \frac{3}{4}(Q+2)$.
Moreover, we assume

$$\boxed{\operatorname{div} b \geq 0} \qquad \text{in } \Omega.$$

Kolmogorov type operators

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Remark.

All of our main results still hold true if we replace assumption **(H3)** with the following one:

$$\boxed{f \in L^q(\Omega)} \quad \textit{for some } q > \frac{Q+2}{2}$$
$$\boxed{b \in (L^q(\Omega))^{m_0}} \quad \textit{for some } q > Q+2 \textit{ in } \Omega.$$

Assumptions

Hypoellipticity of $\mathscr{K}$

- The hypoellipticity of $\mathscr{K}$ is implied by

$$\text{rank Lie}\,(X_1, \ldots, X_m, Y)\,(x,t) = N+1, \qquad \forall\,(x,t) \in \mathbb{R}^{N+1},$$

where we rewrite $\mathscr{K}$ in terms of vector fields as follows

$$\mathscr{K} = \sum_{i=1}^{m} X_i^2 + Y,$$

$$X_i := \sum_{j=1}^{m} \overline{a}_{ij} \partial_{x_j}, \quad i = 1, \ldots, m, \qquad Y := \langle Bx, D \rangle - \partial_t.$$

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- Hörmander constructed the fundamental solution of $\mathscr{K}$ as

$$\Gamma(x,t)=\frac{(4\pi)^{-\frac{N}{2}}}{\sqrt{\det C(t)}}\exp\Big(-\frac{1}{4}\langle C^{-1}(t)x,x\rangle-t\,\mathrm{tr}(B)\Big),\quad t>0.$$

Preliminaries

Lie Group

Given

$$(x,t) \circ (\xi,\tau) = (\xi + E(\tau)x, t+\tau), \quad E(t) := \exp(-tB),$$

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$\mathbb{K}=(\mathbb{R}^{N+1},\circ)$ is a Lie group with inverse

$$(x,t)^{-1}=(-E(-t)x,-t).$$

$\mathscr{K}$ is invariant with respect to the left translation

$$\ell_\zeta:\mathbb{R}^{N+1}\to\mathbb{R}^{N+1},\quad \ell_\zeta(z)=\zeta\circ z.$$

Structural assumptions on B

If B takes the form

$$B_0 = \begin{pmatrix} \mathbb{O} & \mathbb{O} & \dots & \mathbb{O} & \mathbb{O} \\ B_1 & \mathbb{O} & \dots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & B_2 & \dots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & B_\kappa & \mathbb{O} \end{pmatrix}$$

then $\mathscr{K}$ is **hypoelliptic** and **invariant** with respect to $(\delta_r)_{r>0}$

$$\boxed{\mathscr{K}(u \circ \delta_r) = r^2 \delta_r (\mathscr{K} u), \quad \text{for every} \quad r > 0},$$

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$$\boxed{\mathscr{K}\left(u \circ \delta_r\right) = r^2 \delta_r \left(\mathscr{K} u\right), \quad \text{for every} \quad r > 0},$$

where

$$\delta_r := \mathrm{diag}(r I_m, r^3 I_{m_1}, \dots, r^{2\kappa+1} I_{m_\kappa}, r^2).$$

Example: kinetic operator

The simplest example is the kinetic operator is defined for $(x, t) = (v, y, t) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$ as

$$\boxed{\mathscr{K}_0 := \sum_{j=1}^{m} \partial^2_{x_j} - \sum_{j=1}^{m} x_j \partial_{x_{m+j}} - \partial_t = \Delta_v - \langle v, D_y \rangle - \partial_t.}$$

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In this case, the matrix B takes the form

$$B = \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ -I_m & \mathbb{O}, \end{pmatrix}$$

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the group law and the dilations are given by

$$(v,y,t) \circ (v_0,y_0,t_0) := (v_0 + v, y_0 + y + tv_0, t_0 + t),$$
$$\delta_r(v,y,t) := (rv, r^3 y, r^2 t)$$

Kinetic operator: functional space

- D. Albritton, S. Armstrong, J.C. Mourrat and M. Novack, *Variational methods for the kinetic Fokker-Planck equation*, arXiv:1902.04037v2, preprint (2021).

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→ Define the space

$$H^1_{hyp}(U) := \{f \in L^2(U; H^1) : v \cdot \nabla_x f - \partial_t f \in L^2(U; H^{-1})\}$$

Kolmogorov type operators

Weak solutions

We let $\mathcal{W}$ denote the closure of $C^\infty(\overline{\Omega})$ in the norm

$$\|u\|^2_{\mathcal{W}} = \|u\|^2_{L^2(\Omega_{N-m_0+1};H^1_{x^{(0)}})} + \|Yu\|^2_{L^2(\Omega_{N-m_0+1};H^{-1}_{x^{(0)}})}.$$

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Definition

A function $u \in \mathcal{W}$ is a weak solution to $\mathscr{L}u = f$ with source term $f \in L^2(\Omega)$ if for every non-negative test function $\phi \in \mathcal{D}(\Omega)$, we have

$$\int_\Omega -\langle ADu, D\phi\rangle - uY\phi + \langle b, Du\rangle\phi + cu\phi = \int_\Omega f\phi.$$

Contents

Main results

Harnack inequality

Theorem (Harnack inequality, Anceschi Rebucci)

Let $\mathcal{Q}^0 = B_{R_0} \times B_{R_0} \times \ldots \times B_{R_0} \times (-1, 0]$ *and let* u *be a non-negative weak solution to* $\mathscr{L} u = f$ *in* $\Omega \supset \mathcal{Q}^0$ *under assumptions* ***(H1)-(H3)***. *Then we have*

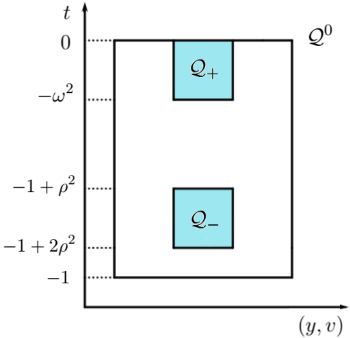
$$\sup_{\mathcal{Q}_-} u \, \leq \, C \left(\inf_{\mathcal{Q}_+} u + \| f \|_{L^q(\mathcal{Q}^0)} \right),$$

where $\mathcal{Q}_+ = B_\omega \times B_{\omega^3} \times \ldots \times B_{\omega^{2\kappa+1}} \times (-\omega^2, 0]$ *and* $\mathcal{Q}_- = B_\omega \times B_{\omega^3} \times \ldots \times B_{\omega^{2\kappa+1}} \times (-1+\rho^2, -1+2\rho^2]$, *with* $0 < \rho < \frac{\omega}{\sqrt{2}}$. *Moreover, the constants* C, ω *and* R_0 *only depend on the homogeneous dimension* Q, q *and on the ellipticity constants* λ *and* Λ.

Main results

Harnack inequality: geometric setting

Figure: Geometric setting of the Harnack inequality for degenerate Kolmogorov type operators



Main results

Hölder continuity

Theorem (Hölder regularity, Anceschi Rebucci)

*There exists $\alpha \in (0,1)$ only depending on dimension Q, λ, Λ such that all weak solutions u to $\mathscr{L}u = f$ under assumption **(H1)- (H3)** in $\Omega \supset \mathcal{Q}_1$ satisfy*

$$[u]_{C^\alpha(Q_{\frac{1}{2}})} \leq C \left(\|u\|_{L^2(\mathcal{Q}_1)} + \|f\|_{L^q(\mathcal{Q}_1)} \right),$$

where the constant C only depends on the homogeneous dimension Q, q and the ellipticity constants λ, Λ.

Comparison with previous results

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Fokker-Planck operator

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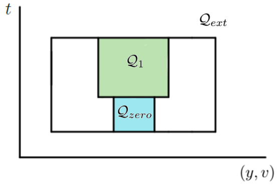
Weak Poincaré inequality: geometric setting

Theorem (Weak Poincaré inequality)

Let $\eta \in (0,1)$; then there exist $R > 1$ and $\vartheta_0 \in (0,1)$ such that for any non-negative function $u \in \mathcal{W}$ such that $u \le M$ in $\mathcal{Q}_1$ for a positive constant M and $|\{u = 0\} \cap \mathcal{Q}_{zero}| \ge \frac{1}{4} |\mathcal{Q}_{zero}|$, we have

$$\|(u - \vartheta_0 M)_+\|_{L^2(\mathcal{Q}_1)} \le C \left(\|D_{m_0} u\|_{L^2(\mathcal{Q}_{ext})} + \|Y u\|_{L^2 H^{-1}(\mathcal{Q}_{ext})} \right),$$

where $C > 0$ is a constant only depending on Q.



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- Weak Poincarè inequality
- Weak expansion of positivity of super-solutions
  - → Covering argument.

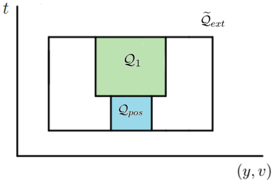
Expansion of positivity of super-solutions

Lemma

*Let $\theta \in (0, 1]$. Then there exist a small positive constant $\eta_0 = \eta_0(\theta, Q, \lambda, \Lambda) \in (0, 1)$ such that for any non-negative weak super-solution u under assumptions **(H1)-(H3)** in $\Omega \supset \widetilde{\mathcal{Q}}_{ext}$ such that*

$$|\{u \geq 1\} \cap \mathcal{Q}_{pos}| \geq \frac{1}{2} |\mathcal{Q}_{pos}|,$$

we have $u \geq \eta_0$ in $\mathcal{Q}_1$.



Weak Harnack inequality

Theorem (Weak Harnack inequality)

*Let $\mathcal{Q}^0 = B_{R_0} \times B_{R_0} \times \ldots \times B_{R_0} \times (-1, 0]$ and let u be a non-negative weak super-solution to $\mathscr{L} u = f$ in $\Omega \supset \mathcal{Q}^0$ under the assumptions **(H1)-(H3)**. Then we have*

$$\left(\int_{\mathcal{Q}_-} u^p \right)^{\frac{1}{p}} \leq C \left(\inf_{\mathcal{Q}_+} u + \| f \|_{L^q(\mathcal{Q}^0)} \right),$$

where $\mathcal{Q}_+ = B_\omega \times B_{\omega^3} \times \ldots \times B_{\omega^{2\kappa+1}} \times (-\omega^2, 0]$ and $\mathcal{Q}_- = B_\omega \times B_{\omega^3} \times \ldots \times B_{\omega^{2\kappa+1}} \times (-1, -1 + \omega^2]$. Moreover, the constants C, p, ω and R_0 only depend on the homogeneous dimension Q, q and on the ellipticity constants λ and Λ.

Bibliography

- F. Anceschi, and A. Rebucci, *A note on the weak regularity theory for degenerate Kolmogorov equations*, preprint arXiv:2107.04441 **[math.AP]** (2021).
- F. Anceschi, and A. Rebucci, *On the fundamental solution for degenerate Kolmogorov equations with rough coefficients*, accepted for publication in *Journal of Elliptic and Parabolic equations* (2022).
- J. Guerand and C. Imbert, *Log-transform and the weak Harnack inequality for kinetic Fokker-Planck equations*, *arXiv: 2102.04105*, preprint, 2021.
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Thank you for your attention

The first order terms

If we consider operator

$$\boxed{\mathscr{L}u := \mathrm{div}(A(z)Du(z)) + Yu(z) + \langle b, Du(z)\rangle + cu(z) = f(z)},$$

with $c \in L^q(\Omega)$ for some $q > \frac{3}{4}(Q+2)$. In this case we obtain the same statement for the weak Harnack inequality if we restric ourselves to the case where $c \geq 0$.
Otherwise we only obtain the statement for weak solutions.

Ink-Spots Theorem

Theorem

Let $E \subset F$ be two bounded measurable sets. We assume there exists a constant $\mu \in]0,1[$ such that

- $E \subset \mathcal{Q}_1$ *and* $|E| < (1-\mu)|\mathcal{Q}_1|$;
- *moreover, there exist an integer m such that for any cylinder $\mathcal{Q} \subset \mathcal{Q}_1$ such that $\overline{\mathcal{Q}}^m \subset \mathcal{Q}_1$ and $|\mathcal{Q} \cap E| \geq (1-\mu)|\mathcal{Q}|$, we have that $\overline{\mathcal{Q}}^m \subset F$.*

Then for some universal constant $c_{\mathrm{is}} \in (0,1)$ only depending on N, there holds

$$|E| \leq \frac{m+1}{m}(1 - c_{\mathrm{is}}\mu)|F|.$$

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- C. Imbert and L Silvestre. *Global regularity estimates for the Boltzmann equation without cut-off* arXiv, preprint (2019).

Assumptions

Hypoellipticity of $\mathscr{K}$

The Hörmander's condition is equivalent to requiring that the matrix B has the form

$$B = \begin{pmatrix} * & * & \ldots & * & * \\ B_1 & * & \ldots & * & * \\ \mathbb{O} & B_2 & \ldots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \ldots & B_\kappa & * \end{pmatrix}$$

where every block B_j is an $m_j \times m_{j-1}$ matrix of rank m_j with $j = 1, 2, \ldots, \kappa$; the m_js are positive integers such that

$$m_0 \geq m_1 \geq \ldots \geq m_\kappa \geq 1, \quad \text{and} \quad m_0 + m_1 + \ldots + m_\kappa = N.$$

Stacked cylinders

Theorem

consider any non-empty cylinder $\mathcal{Q}_r(z_0) \subset \mathcal{Q}_-$ *and we set* $T_k = \sum_{j=1}^{k}(2^k r)^2$. *Let* $N \geq 1$ *such that* $T_N \leq -t_0 < T_{N+1}$ *and let*

$$\mathcal{Q}_r[k] := \mathcal{Q}_{2^k r}(z_k), \quad k = 1, \ldots, N$$
$$\mathcal{Q}_{R_{N+1}}[N+1] := \mathcal{Q}_{R_{N+1}}(z_{N+1}),$$

where $z_k = z_0 \circ (0, \ldots, 0, T_k)$ *and* $R = |t_0 + T_N|^{\frac{1}{2}}$, $R_{N+1} = \max(R, \rho)$, *and*

$$z_{N+1} = \{ \ z_N \circ (0, \ldots, (0, 0),$$

These cylinders satisfy

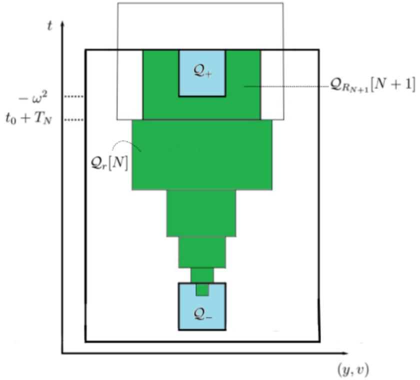
$$\mathcal{Q}_+ \subset \mathcal{Q}_{R_{N+1}}[N+1], \qquad \cup_{k=1}^{N+1} \mathcal{Q}_r[k] \subset (-1, 0] \times B_2, \qquad \widetilde{\mathcal{Q}}[N] \subset \mathcal{Q}_r[N],$$

where $\widetilde{\mathcal{Q}}[N] = \mathcal{Q}_{R_{N+1}/2}\left(z_{N+1} \circ (0, \ldots, 0, -R_{N+1}^2)\right)$.

Main results

Expansion of positivity: geometric setting

Figure: Stacking cylinders above an initial one contained in $\mathcal{Q}_-$



Hölder continuous functions

Definition

Let α be a positive constant, $\alpha \le 1$, and let Ω be an open subset of $\mathbb{R}^{N+1}$. We say that a function $f : \Omega \longrightarrow \mathbb{R}$ is Hölder continuous with exponent α, $f \in C^{\alpha}_{K}(\Omega)$ if there exists a positive constant $C > 0$ such that

$$|f(z) - f(\zeta)| \le C\, d(z,\zeta)^{\alpha} \qquad \text{for every } z, \zeta \in \Omega,$$

where d is the intrinsic distance.
To every bounded function $f \in C^{\alpha}_{K}(\Omega)$ we associate the semi-norm

$$[f]_{C^{\alpha}(\Omega)} = \sup_{\substack{z,\zeta\in\Omega \\ z\neq\zeta}} \frac{|f(z) - f(\zeta)|}{d(z,\zeta)^{\alpha}}.$$

Sobolev type inequality

Theorem (Sobolev Type Inequality for sub-solutions)

*Let **(H1)-(H3)** hold. Let v be a non-negative weak sub-solution of $\mathscr{L}v = f$ in $\mathcal{Q}_1$. Then there exists a constant $C = C(Q, \lambda, \Lambda) > 0$ such that $v \in L^{2\alpha}(\mathcal{Q}_1)$, and the following inequality holds*

$$\| v \|_{L^{2\alpha}(\mathcal{Q}_\rho(z_0))} \leq C \cdot \left(\| b \|_{L^q(\mathcal{Q}_r(z_0))} + \frac{r - \rho + 1}{r - \rho} \right) \| D_{m_0} v \|_{L^2(\mathcal{Q}_r(z_0))} + \\ + C \cdot \left(\| c \|_{L^q(\mathcal{Q}_r(z_0))} + \frac{\rho + 1}{\rho(r - \rho)} \right) \| v \|_{L^2(\mathcal{Q}_r(z_0))} + \\ + C \cdot \| f \|_{L^2(\mathcal{Q}_r(z_0))}$$

for every ρ, r with $\frac{1}{2} \leq \rho < r \leq 1$ and for every $z_0 \in \Omega$. The same statement holds for non-negative super-solutions.

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- $\alpha = \frac{q(Q+2)}{q(Q-2)+2(Q+2)} > 1 \quad \Longleftrightarrow \quad q > \frac{Q+2}{2}$

Caccioppoli type inequality

Theorem (Caccioppoli type inequality for sub-solutions)

*Let **(H1)**-**(H3)** hold. Let r, ρ be such that $\frac{1}{2} \leq \rho < r \leq 1$. Then for every weak sub-solutions to $\mathscr{L} u = f$ we have that for every $p > 1/2$ it holds*

$$\| D_{m_0} v \|^2_{L^2(\mathcal{Q}_\rho)} \leq \frac{4p}{\lambda(2p-1)} \left(\frac{2p}{2p-1} \frac{c_1^2 \Lambda}{(r-\rho)^2} + \frac{c_0}{\rho(r-\rho)} + \frac{\| b \|^2_{L^q(\mathcal{Q}_r)}}{2\epsilon_b} + p \| c \|_{L^q(\mathcal{Q}_r)} + p \| f \|_{L^q(\mathcal{Q}_r)} \right) \| u^p \|^2_{L^{2\beta}(\mathcal{Q}_r)}$$

where $\epsilon_b = \frac{|2p-1|\lambda}{2|p|}$ and c_0, c_1 are constants that only depend on B.

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where $\epsilon_b = \frac{|2p-1|\lambda}{2|p|}$ and c_0, c_1 are constants that only depend on B.

- $\alpha = \frac{q(Q+2)}{q(Q-2)+2(Q+2)} > \beta = \frac{q}{q-1} \qquad \Longleftrightarrow \qquad q > \frac{3}{4}(Q+2)$

Local boundedness for weak sub-solutions

Theorem (Local boundedness of weak sub-solutions)

*Let u be a non-negative weak sub-solution to $\mathscr{L}u = f$ in Ω under the assumptions **(H1)-(H3)**. Let $z_0 \in \Omega$ and r, ρ, with $0 < \rho < r$, be such that $\overline{\mathcal{Q}_r(z_0)} \subseteq \Omega$. Then for every $p \in \mathbb{R}$ such that $p > 1/2$ there exist a positive constant C such that*

$$\sup_{\mathcal{Q}_\rho(z_0)} u^p \leq \frac{C}{(r-\rho)^{4\mu}} \parallel u^p \parallel_{L^\beta(\mathcal{Q}_r)},$$

where

$$\mu := \frac{\alpha}{\alpha - \beta}, \quad \alpha := \frac{q(Q+2)}{q(Q-2) + 2(Q+2)}, \quad \beta := \frac{q}{q-1}.$$

Local boundedness for weak sub-solutions

Comparison with the uniformly parabolic case

- It would be natural to expect that the optimal lower bound for q is $\frac{Q+2}{2}$.

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We represent a solution u to $\mathscr{L} u = f$ in terms of Γ

$$u(x,t) = \int_{\Omega} \Gamma(x,t;\xi,\tau) \mathscr{K} u(\xi,\tau) d\xi d\tau$$

Thus, we get exactly estimate with the following additional term regarding the source term f:

$$\boxed{|p| \int_{\mathcal{Q}_r} |f| u^{2p-1} \psi^2}_F$$

We therefore need to properly estimate u^{2p-1}. In particular, if

- $0 \leq u < 1$ we have that $u^{2p-1} < 1$, thus

$$\boxed{|p| \int_{\mathcal{Q}_r} |f| u^{2p-1} \psi^2}_F \leq |p| \int_{\mathcal{Q}_r} |f| \psi^2 \tag{1}$$

- $u \geq 1$ we have that $u^{2p-1} < u^p = v$. By combining Hölder's inequality and Young's inequality for every $\epsilon > 0$ we get

$$\boxed{|p| \int_{\mathcal{Q}_r} |f| u^{2p-1} \psi^2}_F \leq |p| \|f\|_{L^2(\mathcal{Q}_r)} \|v\|_{L^2(\mathcal{Q}_r)} \tag{2}$$

We now address the case $p > 1$. Reasoning as above, we set $v_{n,p} = g_{n,p}(u)$ and we choose the test function as in (**??**). Then the weak formulation of (**??**) for $v_{n,p}$ reads exactly as (**??**) with the additional term

$$\boxed{-p \int\limits_{\mathcal{Q}_r} f v_{n,p}(u)\, v'_{n,p}(u)\psi^2}_F \quad .$$