Kolmogorov operators on noncompact metric graphs

Abdelaziz Rhandi
(University of Salerno)

"Kolmogorov Operators and their Applications", Cortona
13-17 June, 2022
A joint work with Delio Mugnolo (Univ. Hagen).

Metric graphs:

Metric graphs:

Consider countable sets I, J and $(a_j)_{j \in J} \subset [0, \infty]$. Define

Metric graphs:

Consider countable sets I, J and $(a_j)_{j\in J} \subset [0,\infty]$. Define

$$\mathcal{E} := \bigsqcup_{i\in I} [0,\infty) \sqcup \bigsqcup_{j\in J} [0,a_j].$$

Metric graphs:

Consider countable sets I, J and $(a_j)_{j\in J} \subset [0,\infty]$. Define

$$\mathcal{E} := \bigsqcup_{i\in I}[0,\infty) \sqcup \bigsqcup_{j\in J}[0,a_j].$$

$$\mathcal{V} := \bigsqcup_{i\in I}\{0\} \sqcup \bigsqcup_{j\in J}\{0,a_j\}.$$

On $\mathcal{E}$ define the metric

$$d_{\mathcal{E}}((x,i),(y,k)) := \begin{cases} |x-y|, & \text{if } i = k \text{ and } x,y \in [0,a_i] (\text{or } [0,\infty)), \\ \infty, & \text{otherwise }, \end{cases}$$

Metric graphs:

Consider countable sets I, J and $(a_j)_{j\in J} \subset [0,\infty]$. Define

$$\mathcal{E} := \bigsqcup_{i\in I}[0,\infty) \sqcup \bigsqcup_{j\in J}[0,a_j].$$

$$\mathcal{V} := \bigsqcup_{i\in I}\{0\} \sqcup \bigsqcup_{j\in J}\{0,a_j\}.$$

On $\mathcal{E}$ define the metric

$$d_{\mathcal{E}}((x,i),(y,k)) := \begin{cases} |x-y|, & \text{if } i=k \text{ and } x,y\in[0,a_i](\text{or } [0,\infty)), \\ \infty, & \text{otherwise }, \end{cases}$$

Equivalence relation: $(x,i)\sim(y,k)$ iff either $x=y$ and $i=k$ or else $x=y=0$ regardless of i, k.

Metric graphs:

Consider countable sets I, J and $(a_j)_{j\in J} \subset [0,\infty]$. Define

$$\mathcal{E} := \bigsqcup_{i\in I}[0,\infty) \sqcup \bigsqcup_{j\in J}[0,a_j].$$

$$\mathcal{V} := \bigsqcup_{i\in I}\{0\} \sqcup \bigsqcup_{j\in J}\{0,a_j\}.$$

On $\mathcal{E}$ define the metric

$$d_{\mathcal{E}}((x,i),(y,k)) := \begin{cases} |x-y|, & \text{if } i=k \text{ and } x,y \in [0,a_i](\text{or } [0,\infty)), \\ \infty, & \text{otherwise }, \end{cases}$$

Equivalence relation: $(x,i) \sim (y,k)$ iff either $x=y$ and $i=k$ or else $x=y=0$ regardless of i, k.
A metric graph is $\mathcal{G} := \mathcal{E}/\sim$.

Second-order elliptic operators on metric graphs:

Second-order elliptic operators on metric graphs:

The theory of second-order differential operators on compact metric graphs of the form $\mathcal{G}_c := \bigsqcup_{j\in J}[0, a_j]/\sim$:

Second-order elliptic operators on metric graphs:

The theory of second-order differential operators on compact metric graphs of the form $\mathcal{G}_c := \bigsqcup_{j\in J}[0, a_j]/\sim$:

- G. Lumer in 1979 and Pavlov and Faddeev in 1983.

Second-order elliptic operators on metric graphs:

The theory of second-order differential operators on compact metric graphs of the form $\mathcal{G}_c := \bigsqcup_{j\in J}[0, a_j]/\sim$:

- G. Lumer in 1979 and Pavlov and Faddeev in 1983.
- J.P. Roth in 1983: Explizit formula of the heat semigroup generated by Δ with continuity and Kirchhoff-type boundary conditions.

Second-order elliptic operators on metric graphs:

The theory of second-order differential operators on compact metric graphs of the form $\mathcal{G}_c := \bigsqcup_{j\in J}[0, a_j]/\sim$:

- G. Lumer in 1979 and Pavlov and Faddeev in 1983.
- J.P. Roth in 1983: Explizit formula of the heat semigroup generated by Δ with continuity and Kirchhoff-type boundary conditions.
- S. Nicaise in 1984 extended the results of Roth, see also C. Cattaneo in 1999 and many others...

Kolmogorov operators on star graphs

Kolmogorov operators on star graphs

Consider

$$\mathcal{S}_m := \bigsqcup_{i=1}^{m} [0,\infty)\big/\sim.$$

Kolmogorov operators on star graphs

Consider

$$\mathcal{S}_m := \bigsqcup_{i=1}^{m} [0,\infty)\big/\sim.$$

Notations: $0 := (0,i)$ for all i, $x_i := (x,i)$ and $|x_i| := x$ whenever $x \geq 0$.

Kolmogorov operators on star graphs

Consider

$$\mathcal{S}_m := \bigsqcup_{i=1}^{m} [0,\infty)\big/\sim .$$

Notations: $0 := (0,i)$ for all i, $x_i := (x,i)$ and $|x_i| := x$ whenever $x \geq 0$.

On $C_b(\mathcal{S}_m)$,

Kolmogorov operators on star graphs

Consider

$$\mathcal{S}_m := \bigsqcup_{i=1}^{m} [0,\infty)\big/\sim .$$

Notations: $0 := (0, i)$ for all i, $x_i := (x, i)$ and $|x_i| := x$ whenever $x \geq 0$.

On $C_b(\mathcal{S}_m)$,

$$Lf(x_i) = q(|x_i|)f''(x_i) + b(|x_i|)f'(x_i) + c(|x_i|)f(x_i),\ |x_i| \geq 0,\ i = 1, \ldots, m,$$

Kolmogorov operators on star graphs

Consider

$$\mathcal{S}_m := \bigsqcup_{i=1}^{m} [0,\infty)\big/\sim .$$

Notations: $0 := (0,i)$ for all i, $x_i := (x,i)$ and $|x_i| := x$ whenever $x \geq 0$.

On $C_b(\mathcal{S}_m)$,

$$Lf(x_i) = q(|x_i|)f''(x_i) + b(|x_i|)f'(x_i) + c(|x_i|)f(x_i),\ |x_i| \geq 0,\ i = 1,\dots,m,$$

with

$$D(L) = \{f \in C_b(\mathcal{S}_m) \cap \bigcap_{1\leq p<\infty} \widetilde{W^{2,p}_{\mathrm{loc}}}(\mathcal{S}_m) : \sum_{i=1}^{m} f'(0_i) = 0,\ Lf \in C_b(\mathcal{S}_m)\},$$

where

$$\widetilde{W^{k,p}_{\mathrm{loc}}}(\mathcal{S}_m) := \bigoplus_{i=1}^{m} W^{k,p}_{\mathrm{loc}}(\mathbb{R}_+).$$

The associated Cauchy problem

The associated Cauchy problem

Aim 1:

The associated Cauchy problem

Aim 1:
Existence and uniqueness of classical solutions to

The associated Cauchy problem

Aim 1:
Existence and uniqueness of classical solutions to

$$\begin{cases} \partial_t u(t,\cdot) = Lu(t,\cdot), \quad t > 0, \\ u(0,\cdot) = f(\cdot), \end{cases} \qquad (\mathrm{P}_L)$$

The associated Cauchy problem

Aim 1:
Existence and uniqueness of classical solutions to

$$\begin{cases} \partial_t u(t,\cdot) = Lu(t,\cdot), & t>0, \\ u(0,\cdot) = f(\cdot), \end{cases} \tag{P_L}$$

Assumptions:
$q, b, c \in C^\alpha_{\mathrm{loc}}([0,\infty))$, $b(0)=0$, $q(x)>0$, $\forall x \in [0,\infty)$ and $\sup c \le c_0$ for some $c_0 \in \mathbb{R}$.

Existence of classical solutions

Existence of classical solutions

Main idea: Compare (P_L) with $(\mathrm{P}_{\tilde{L}})$, where

Existence of classical solutions

Main idea: Compare (P_L) with $(\mathrm{P}_{\tilde{L}})$, where

$$\begin{aligned}\tilde{L}f(x) &= \tilde{q}(x)f''(x) + \tilde{b}(x)f'(x) + \tilde{c}(x)f(x)\\ D(\tilde{L}) &= \{f \in C_b(\mathbb{R}) \cap \bigcap_{1\le p<\infty} W^{2,p}_{\mathrm{loc}}(\mathbb{R}) : \tilde{L}f \in C_b(\mathbb{R})\},\end{aligned}$$

Existence of classical solutions

Main idea: Compare (P_L) with $(\mathrm{P}_{\tilde{L}})$, where

$$\begin{aligned} \tilde{L}f(x) &= \tilde{q}(x)f''(x) + \tilde{b}(x)f'(x) + \tilde{c}(x)f(x) \\ D(\tilde{L}) &= \{f \in C_b(\mathbb{R}) \cap \bigcap_{1\le p<\infty} W^{2,p}_{\mathrm{loc}}(\mathbb{R}) : \tilde{L}f \in C_b(\mathbb{R})\}, \end{aligned}$$

$\tilde{q}(x) = q(x)$, $\tilde{b}(x) = b(x)$, $\tilde{c}(x) = c(x)$ if $x \ge 0$ and
$\tilde{q}(x) = q(-x)$, $\tilde{b}(x) = -b(-x)$, $\tilde{c}(x) = c(-x)$ if $x \le 0$.

Existence of classical solutions

Existence of classical solutions

Important remark:

Every $f \in C_b(\mathcal{S}_m)$ uniquely determines m functions $\tilde{f}_i \in C_b(\mathbb{R})$

$$\tilde{f}_i(x) := \begin{cases} f(x_i), & |x_i| = x, \quad \text{if } x \geq 0, \\ \frac{2}{m} \sum_{1 \leq j \leq m} f(-x_j) - f(-x_i), & |x_i| = -x, \quad \text{if } x \leq 0. \end{cases}$$

Existence of classical solutions

Important remark:

Every $f \in C_b(\mathcal{S}_m)$ uniquely determines m functions $\tilde{f}_i \in C_b(\mathbb{R})$

$$\tilde{f}_i(x) := \begin{cases} f(x_i), & |x_i| = x, \quad \text{if } x \geq 0, \\ \frac{2}{m} \sum_{1\leq j\leq m} f(-x_j) - f(-x_i), & |x_i| = -x, \quad \text{if } x \leq 0. \end{cases}$$

There is a unique classical solution
$u_i^n \in C([0,\infty) \times (-n,n)) \cap C_{loc}^{1+\frac{\alpha}{2},2+\alpha}((0,\infty) \times [-n,n])$,
$1 \leq i \leq m$, to the Cauchy–Dirichlet problem

Existence of classical solutions

Important remark:
Every $f \in C_b(\mathcal{S}_m)$ uniquely determines m functions $\tilde{f}_i \in C_b(\mathbb{R})$

$$\tilde{f}_i(x) := \begin{cases} f(x_i), & |x_i| = x, \quad \text{if } x \geq 0, \\ \frac{2}{m} \sum\limits_{1 \leq j \leq m} f(-x_j) - f(-x_i), & |x_i| = -x, \quad \text{if } x \leq 0. \end{cases}$$

There is a unique classical solution
$u_i^n \in C([0,\infty) \times (-n,n)) \cap C_{loc}^{1+\frac{\alpha}{2},2+\alpha}((0,\infty) \times [-n,n])$,
$1 \leq i \leq m$, to the Cauchy–Dirichlet problem

$$\begin{cases} \partial_t u_i^n(t,\cdot) = \tilde{L} u_i^n(t,\cdot), & t > 0, \\ u_i^n(t,-n) = u_i^n(t,n) = 0, & t > 0, \\ u_i^n(0,x) = \tilde{f}_i(x), & x \in (-n,n). \end{cases}$$

Existence of classical solutions

Existence of classical solutions

One proves

$$\hat{u}^n(t, x_i) := u_i^n(t, |x_i|),\ i = 1, \ldots, m,\ |x_i| \geq 0,\ t \geq 0$$

is the classical solution of problem (P_L) on $\mathcal{S}_m^n := \bigsqcup_{i=1}^m [0, n] \big/ \sim$.

Existence of classical solutions

One proves

$$\hat{u}^n(t,x_i) := u_i^n(t,|x_i|),\ i=1,\dots,m,\ |x_i|\ge 0,\ t\ge 0$$

is the classical solution of problem (P_L) on $\mathcal{S}_m^n := \bigsqcup_{i=1}^m [0,n]/\sim$.
Since one knows

$$T(t)\tilde{f}_i(x) := \lim_{n\to\infty} u_i^n(t,x)$$

exists for $t\ge 0$, $x\in\mathbb{R}$, belongs to
$C([0,\infty)\times\mathbb{R})\cap C_{\mathrm{loc}}^{1+\frac{\alpha}{2},2+\alpha}((0,\infty)\times\mathbb{R})$ and is a classical solution
$(\mathrm{P}_{\tilde{L}})$ on $(0,\infty)\times\mathbb{R}$ with initial data $\tilde{f}_i$,

Existence of classical solutions

One proves

$$\hat{u}^n(t,x_i) := u_i^n(t,|x_i|),\ i=1,\ldots,m,\ |x_i|\geq 0,\ t\geq 0$$

is the classical solution of problem (P_L) on $\mathcal{S}_m^n := \bigsqcup_{i=1}^m [0,n]\diagup\sim$.
Since one knows

$$T(t)\tilde{f}_i(x) := \lim_{n\to\infty} u_i^n(t,x)$$

exists for $t\geq 0$, $x\in\mathbb{R}$, belongs to
$C([0,\infty)\times\mathbb{R})\cap C_{\mathrm{loc}}^{1+\frac{\alpha}{2},2+\alpha}((0,\infty)\times\mathbb{R})$ and is a classical solution ($\mathrm{P}_{\tilde{L}}$) on $(0,\infty)\times\mathbb{R}$ with initial data $\tilde{f}_i$,
one deduces that

Existence of classical solutions

One proves

$$\hat{u}^n(t,x_i) := u_i^n(t,|x_i|),\ i=1,\dots,m,\ |x_i|\geq 0,\ t\geq 0$$

is the classical solution of problem (P_L) on $\mathcal{S}_m^n := \bigsqcup_{i=1}^m [0,n]\big/\sim$.
Since one knows

$$T(t)\tilde{f}_i(x) := \lim_{n\to\infty} u_i^n(t,x)$$

exists for $t\geq 0$, $x\in\mathbb{R}$, belongs to
$C([0,\infty)\times\mathbb{R})\cap C_{\text{loc}}^{1+\frac{\alpha}{2},2+\alpha}((0,\infty)\times\mathbb{R})$ and is a classical solution
$(\mathrm{P}_{\tilde{L}})$ on $(0,\infty)\times\mathbb{R}$ with initial data $\tilde{f}_i$,
one deduces that $T_m(t)f(x_i) := T(t)\tilde{f}_i(|x_i|)$ is a classical solution to (P_L).

Heat kernel

Heat kernel

Since

$$T(t)f(x) = \int_{\mathbb{R}} k(t,x,y)f(y)\,dy,\ f \in C_b(\mathbb{R}),\ t > 0,\ x \in \mathbb{R},$$

it follows that (using the definition of $\tilde{f}_i$)

Heat kernel

Since

$$T(t)f(x) = \int_{\mathbb{R}} k(t,x,y)f(y)\,dy,\ f \in C_b(\mathbb{R}),\ t > 0,\ x \in \mathbb{R},$$

it follows that (using the definition of $\tilde{f}_i$)

$$T_m(t)f(x_i) = \int_{(\mathbb{R}_+,i)} \big(k(t,|x_i|,|y_i|) - k(t,|x_i|,-|y_i|)\big) f(y_i)\,dy_i + \sum_{j=1}^{m} \int_{(\mathbb{R}_+,j)} \frac{2}{m} k(t,|x_i|,-|y_j|) f(y_j)\,dy_j.$$

The Ornstein-Uhlenbeck operator on $C_b(\mathcal{S}_m)$

The Ornstein-Uhlenbeck operator on $C_b(\mathcal{S}_m)$

$$Af(x_i) = \frac{1}{2}f''(x_i) - |x_i|f'(x_i),\ |x_i| \geq 0,\ i = 1, \ldots, m,$$

with Kirchhoff-type condition in zero encoded in the domain

The Ornstein-Uhlenbeck operator on $C_b(\mathcal{S}_m)$

$$Af(x_i) = \frac{1}{2} f''(x_i) - |x_i| f'(x_i),\ |x_i| \geq 0,\ i = 1, \ldots, m,$$

with Kirchhoff-type condition in zero encoded in the domain

$$D(A) = \{ f \in C_b(\mathcal{S}_m) \cap \bigcap_{1 \leq p < \infty} \widetilde{W^{2,p}_{\text{loc}}}(\mathcal{S}_m) : \sum_{i=1}^{m} f'(0_i) = 0,\ Af \in C_b(\mathcal{S}_m) \}.$$

The Ornstein-Uhlenbeck operator on $C_b(\mathcal{S}_m)$

$$Af(x_i) = \frac{1}{2}f''(x_i) - |x_i|f'(x_i),\ |x_i| \geq 0,\ i = 1, \dots, m,$$

with Kirchhoff-type condition in zero encoded in the domain

$$D(A) = \{f \in C_b(\mathcal{S}_m) \cap \bigcap_{1\leq p<\infty} \widetilde{W^{2,p}_{\text{loc}}}(\mathcal{S}_m) : \sum_{i=1}^{m} f'(0_i) = 0,\ Af \in C_b(\mathcal{S}_m)\}.$$

Remark:

- $m = 1$: Ornstein-Uhlenbeck operator on $[0, \infty)$ with Neumann boundary condition.

The Ornstein-Uhlenbeck operator on $C_b(\mathcal{S}_m)$

$$Af(x_i) = \frac{1}{2} f''(x_i) - |x_i| f'(x_i),\ |x_i| \geq 0,\ i = 1, \ldots, m,$$

with Kirchhoff-type condition in zero encoded in the domain

$$D(A) = \{f \in C_b(\mathcal{S}_m) \cap \bigcap_{1 \leq p < \infty} \widetilde{W^{2,p}_{\text{loc}}}(\mathcal{S}_m) : \sum_{i=1}^{m} f'(0_i) = 0,\ Af \in C_b(\mathcal{S}_m)\}.$$

Remark:

- $m = 1$: Ornstein-Uhlenbeck operator on $[0, \infty)$ with Neumann boundary condition.
- $m = 2$: Ornstein-Uhlenbeck operator on $\mathbb{R}$.

The associated Ornstein-Uhlenbeck semigroup

The associated Ornstein-Uhlenbeck semigroup

From (1) one can see that associated semigroup is

The associated Ornstein-Uhlenbeck semigroup

From (1) one can see that associated semigroup is

$$(S_m(t)f)(x_i) := \frac{1}{\sqrt{\pi(1-e^{-2t})}} \int_{(\mathbb{R}_+,i)} \Big(\exp[-(1-e^{-2t})^{-1}(e^{-t}|x_i| - |y_i|)^2] - \exp[-(1-e^{-2t})^{-1}(e^{-t}|x_i| + |y_i|)^2] \Big) f(y_i)\, dy_i$$

$$+ \frac{2}{m\sqrt{\pi(1-e^{-2t})}} \times \sum_{1\leq j\leq m} \int_{(\mathbb{R}_+,j)} \exp[-(1-e^{-2t})^{-1}(e^{-t}|x_i| + |y_j|)^2] f(y_j)\, dy_j.$$

The associated invariant measure

The associated invariant measure

The invariant measure of the OU-semigroup on $\mathbb{R}$ is

The associated invariant measure

The invariant measure of the OU-semigroup on $\mathbb{R}$ is

$$\mu(dx) = \frac{1}{\sqrt{\pi}} e^{-|x|^2} dx.$$

The associated invariant measure

The invariant measure of the OU-semigroup on $\mathbb{R}$ is

$$\mu(dx) = \frac{1}{\sqrt{\pi}} e^{-|x|^2} dx.$$

Using the definition of $\tilde{f}_i$ one deduces

The associated invariant measure

The invariant measure of the OU-semigroup on $\mathbb{R}$ is

$$\mu(dx) = \frac{1}{\sqrt{\pi}} e^{-|x|^2} dx.$$

Using the definition of $\tilde{f}_i$ one deduces

$$\mu_m(dx_i) = \frac{2}{m\sqrt{\pi}} e^{-|x_i|^2} dx_i,\ i = 1, \ldots, m,$$

is the unique invariant measure for $S_m(\cdot)$, i.e.

The associated invariant measure

The invariant measure of the OU-semigroup on $\mathbb{R}$ is

$$\mu(dx) = \frac{1}{\sqrt{\pi}} e^{-|x|^2} dx.$$

Using the definition of $\tilde{f}_i$ one deduces

$$\mu_m(dx_i) = \frac{2}{m\sqrt{\pi}} e^{-|x_i|^2} dx_i,\ i = 1, \ldots, m,$$

is the unique invariant measure for $S_m(\cdot)$, i.e.

$$\sum_{i=1}^{m} \int_{(\mathbb{R}_+, i)} S_m(t) f(x_i) \mu_m(dx_i) = \sum_{i=1}^{m} \int_{(\mathbb{R}_+, i)} f(x_i) \mu_m(dx_i).$$

The Ornstein-Uhlenbeck operator on $L^2_{\mu_m}(\mathcal{S}_m)$

The Ornstein-Uhlenbeck operator on $L^2_{\mu_m}(\mathcal{S}_m)$

So, $S_m(\cdot)$ can be extended to a compact C_0-semigroup on $L^2_{\mu_m}(\mathcal{S}_m)$ $(= \bigoplus_{i=1}^m L^2_{\mu_m}(\mathbb{R}_+, i))$. Its generator A_2 is the $L^2_{\mu_m}$-realization of A.

The Ornstein-Uhlenbeck operator on $L^2_{\mu_m}(\mathcal{S}_m)$

So, $S_m(\cdot)$ can be extended to a compact C_0-semigroup on $L^2_{\mu_m}(\mathcal{S}_m)$ $(= \bigoplus_{i=1}^m L^2_{\mu_m}(\mathbb{R}_+, i))$. Its generator A_2 is the $L^2_{\mu_m}$-realization of A. Using the theory of forms

The Ornstein-Uhlenbeck operator on $L^2_{\mu_m}(\mathcal{S}_m)$

So, $S_m(\cdot)$ can be extended to a compact C_0-semigroup on $L^2_{\mu_m}(\mathcal{S}_m)$ $(= \bigoplus_{i=1}^m L^2_{\mu_m}(\mathbb{R}_+, i))$. Its generator A_2 is the $L^2_{\mu_m}$-realization of A. Using the theory of forms

$$D(A_2) = \left\{ f \in \widetilde{H^2_{\mu_m}}(\mathcal{S}_m) : f_i(0) = f_j(0), \forall i, j, \sum_{i=1}^m f_i'(0) = 0 \right\}$$

$$(A_2 f)_i(x) = \frac{1}{2} f_i''(x) - x f_i'(x), \; f = (f_i)_{1 \le i \le m} \in D(A_2),$$

The Ornstein-Uhlenbeck operator on $L^2_{\mu_m}(\mathcal{S}_m)$

So, $S_m(\cdot)$ can be extended to a compact C_0-semigroup on $L^2_{\mu_m}(\mathcal{S}_m)$ $(= \bigoplus_{i=1}^m L^2_{\mu_m}(\mathbb{R}_+, i))$. Its generator A_2 is the $L^2_{\mu_m}$-realization of A. Using the theory of forms

$$D(A_2) = \left\{ f \in \widetilde{H^2_{\mu_m}}(\mathcal{S}_m) : f_i(0) = f_j(0),\, \forall i, j,\, \sum_{i=1}^m f_i'(0) = 0 \right\}$$

$$(A_2 f)_i(x) = \frac{1}{2} f_i''(x) - x f_i'(x),\; f = (f_i)_{1 \le i \le m} \in D(A_2),$$

where

$$\widetilde{H^k_{\mu_m}}(\mathcal{S}_m) := \bigoplus_{i=1}^m H^k_{\mu_m}(\mathbb{R}_+, i),\; k \in \mathbb{N}.$$

The spectrum of A_2 for $m = 1, 2$

The spectrum of A_2 for $m = 1, 2$

- **Case m=2:** By Metafune-Pallara-Priola, JFA. 2002,

The spectrum of A_2 for $m = 1, 2$

- **Case m=2:** By Metafune-Pallara-Priola, JFA. 2002,

$$\sigma(A_2) = \sigma_p(A_2) = \{-k : k \in \mathbb{N}_0\}.$$

The spectrum of A_2 for $m = 1, 2$

- **Case m=2:** By Metafune-Pallara-Priola, JFA. 2002,

$$\sigma(A_2) = \sigma_p(A_2) = \{-k : k \in \mathbb{N}_0\}.$$

The corresponding eigenfunctions are the Hermite polynomials

$$H_k(x) := (-1)^k e^{|x|^2} D^k e^{-|x|^2}, \quad x \in \mathbb{R}, \; k \in \mathbb{N}_0.$$

The spectrum of A_2 for $m = 1, 2$

- **Case m=2:** By Metafune-Pallara-Priola, JFA. 2002,

$$\sigma(A_2) = \sigma_p(A_2) = \{-k : k \in \mathbb{N}_0\}.$$

The corresponding eigenfunctions are the Hermite polynomials

$$H_k(x) := (-1)^k e^{|x|^2} D^k e^{-|x|^2}, \quad x \in \mathbb{R},\ k \in \mathbb{N}_0.$$

- **Case m=1:**

The spectrum of A_2 for $m = 1, 2$

- **Case m=2:** By Metafune-Pallara-Priola, JFA. 2002,

$$\sigma(A_2) = \sigma_p(A_2) = \{-k : k \in \mathbb{N}_0\}.$$

The corresponding eigenfunctions are the Hermite polynomials

$$H_k(x) := (-1)^k e^{|x|^2} D^k e^{-|x|^2}, \quad x \in \mathbb{R},\ k \in \mathbb{N}_0.$$

- **Case m=1:**

$$\sigma(A_2) = \sigma_p(A_2) = \{-2k : k \in \mathbb{N}_0\}.$$

The spectrum of A_2 for $m = 1, 2$

- **Case m=2:** By Metafune-Pallara-Priola, JFA. 2002,

$$\sigma(A_2) = \sigma_p(A_2) = \{-k : k \in \mathbb{N}_0\}.$$

The corresponding eigenfunctions are the Hermite polynomials

$$H_k(x) := (-1)^k e^{|x|^2} D^k e^{-|x|^2}, \quad x \in \mathbb{R},\ k \in \mathbb{N}_0.$$

- **Case m=1:**

$$\sigma(A_2) = \sigma_p(A_2) = \{-2k : k \in \mathbb{N}_0\}.$$

H_{2k} are the eigenfunctions.

The spectrum of A_2

[1] even means $f(x_i) = f(x_j)$, $\forall i, j = 1, \ldots, m$, odd means $f(x_1) + \ldots + f(x_m) = 0$.

The spectrum of A_2

Consider the unitary operator on $L^2_{\mu_m}(\mathcal{S}_m)$

$$R : (f_1, \ldots, f_{m-1}, f_m) \mapsto (f_2, \ldots, f_m, f_1).$$

[1] even means $f(x_i) = f(x_j)$, $\forall i, j = 1, \ldots, m$, odd means $f(x_1) + \ldots + f(x_m) = 0$.

The spectrum of A_2

Consider the unitary operator on $L^2_{\mu_m}(\mathcal{S}_m)$

$$R : (f_1, \dots, f_{m-1}, f_m) \mapsto (f_2, \dots, f_m, f_1).$$

Since $RA_2 = A_2 R$, by the spectral theorem, eigenfunctions of $A_2 =$ eigenfunctions of R.

[1] even means $f(x_i) = f(x_j)$, $\forall i, j = 1, \dots, m$, odd means $f(x_1) + \dots + f(x_m) = 0$.

The spectrum of A_2

Consider the unitary operator on $L^2_{\mu_m}(\mathcal{S}_m)$

$$R : (f_1, \ldots, f_{m-1}, f_m) \mapsto (f_2, \ldots, f_m, f_1).$$

Since $RA_2 = A_2 R$, by the spectral theorem, eigenfunctions of $A_2 =$ eigenfunctions of R.
Since $R^m = I$, the eigenvalued of R are
$z^j := e^{\frac{2j\pi i}{m}}$, $j = 0, \ldots, m-1$.

[1] even means $f(x_i) = f(x_j)$, $\forall i, j = 1, \ldots, m$, odd means $f(x_1) + \ldots + f(x_m) = 0$.

The spectrum of A_2

Consider the unitary operator on $L^2_{\mu_m}(\mathcal{S}_m)$

$$R : (f_1, \ldots, f_{m-1}, f_m) \mapsto (f_2, \ldots, f_m, f_1).$$

Since $RA_2 = A_2 R$, by the spectral theorem, eigenfunctions of $A_2 =$ eigenfunctions of R.
Since $R^m = I$, the eigenvalued of R are $z^j := e^{\frac{2j\pi i}{m}}$, $j = 0, \ldots, m-1$.
The corresponding j-th eigenspace is

[1] even means $f(x_i) = f(x_j)$, $\forall i, j = 1, \ldots, m$, odd means $f(x_1) + \ldots + f(x_m) = 0$.

The spectrum of A_2

Consider the unitary operator on $L^2_{\mu_m}(\mathcal{S}_m)$

$$R : (f_1, \dots, f_{m-1}, f_m) \mapsto (f_2, \dots, f_m, f_1).$$

Since $RA_2 = A_2R$, by the spectral theorem, eigenfunctions of $A_2 =$ eigenfunctions of R.
Since $R^m = I$, the eigenvalued of R are
$z^j := e^{\frac{2j\pi i}{m}}$, $j = 0, \dots, m-1$.
The corresponding j-th eigenspace is

$$E_j := (1, z^j, z^{2j}, \dots, z^{j(m-1)}) \otimes L^2_{\mu_m}(\mathbb{R}_+).$$

[1] even means $f(x_i) = f(x_j)$, $\forall i, j = 1, \dots, m$, odd means $f(x_1) + \dots + f(x_m) = 0$.

The spectrum of A_2

Consider the unitary operator on $L^2_{\mu_m}(\mathcal{S}_m)$

$$R : (f_1, \ldots, f_{m-1}, f_m) \mapsto (f_2, \ldots, f_m, f_1).$$

Since $RA_2 = A_2 R$, by the spectral theorem, eigenfunctions of $A_2 =$ eigenfunctions of R.
Since $R^m = I$, the eigenvalued of R are
$z^j := e^{\frac{2j\pi i}{m}}$, $j = 0, \ldots, m-1$.
The corresponding j-th eigenspace is

$$E_j := (1, z^j, z^{2j}, \ldots, z^{j(m-1)}) \otimes L^2_{\mu_m}(\mathbb{R}_+).$$

Note $E_0 = L^2_{\mathrm{even}}$ and $\bigoplus_{j=1}^{m-1} E_j = L^2_{\mathrm{odd}}$.[1]

[1] even means $f(x_i) = f(x_j)$, $\forall i, j = 1, \ldots, m$, odd means $f(x_1) + \ldots + f(x_m) = 0$.

The spectrum of A_2

The spectrum of A_2

Oddness induces, by $0 = \sum_{i=1}^{m} f(0_i) = mf(0)$, Dirichlet boundary conditions at 0.

The spectrum of A_2

Oddness induces, by $0 = \sum_{i=1}^{m} f(0_i) = mf(0)$, Dirichlet boundary conditions at 0.
$\sigma(A_{|E_j}) = \{-2k-1 : k \in \mathbb{N}_0\}$, $\forall j = 1, \ldots, m-1$.

The spectrum of A_2

Oddness induces, by $0 = \sum_{i=1}^{m} f(0_i) = mf(0)$, Dirichlet boundary conditions at 0.
$\sigma(A_{|E_j}) = \{-2k-1 : k \in \mathbb{N}_0\}$, $\forall j = 1, \ldots, m-1$.
Therefore,

Theorem:

The spectrum of A_2

Oddness induces, by $0 = \sum_{i=1}^{m} f(0_i) = mf(0)$, Dirichlet boundary conditions at 0.
$\sigma(A_{|E_j}) = \{-2k-1 : k \in \mathbb{N}_0\}$, $\forall j = 1, \dots, m-1$.
Therefore,

Theorem:

$$\sigma(A_2) = \{-k : k \in \mathbb{N}_0\},$$

where all even eigenvalues have multiplicity 1, whereas all odd eigenvalues have multiplicity $m-1$.

The harmonic oscillator

The harmonic oscillator

Harmonic oscillator:

$$Bf(x_i) = \frac{1}{2}\left(f''(x_i) - |x_i|^2 f(x_i) + f(x_i)\right), \qquad |x_i| \geq 0, \quad i = 1, \ldots, m.$$

The harmonic oscillator

Harmonic oscillator:

$$Bf(x_i) = \frac{1}{2}\left(f''(x_i) - |x_i|^2 f(x_i) + f(x_i)\right), \qquad |x_i| \geq 0, \quad i = 1, \ldots, m.$$

Define the isometry

$$T : L^2_{\mu_m}(\mathcal{S}_m) \to L^2(\mathcal{S}_m)$$
$$f \mapsto (\sqrt{c_m} e^{-\frac{x^2}{2}} f_i),$$

The harmonic oscillator

Harmonic oscillator:

$$Bf(x_i) = \frac{1}{2}\left(f''(x_i) - |x_i|^2 f(x_i) + f(x_i)\right), \qquad |x_i| \geq 0, \quad i = 1, \dots, m.$$

Define the isometry

$$T : L^2_{\mu_m}(\mathcal{S}_m) \to L^2(\mathcal{S}_m)$$
$$f \mapsto (\sqrt{c_m} e^{-\frac{x^2}{2}} f_i),$$

where $c_m := \frac{2}{m\sqrt{\pi}}$. Thus,

The harmonic oscillator

Harmonic oscillator:

$$Bf(x_i) = \frac{1}{2}\left(f''(x_i) - |x_i|^2 f(x_i) + f(x_i)\right), \qquad |x_i| \geq 0, \quad i = 1, \dots, m.$$

Define the isometry

$$T : L^2_{\mu_m}(\mathcal{S}_m) \to L^2(\mathcal{S}_m)$$
$$f \mapsto (\sqrt{c_m} e^{-\frac{x^2}{2}} f_i),$$

where $c_m := \frac{2}{m\sqrt{\pi}}$. Thus,

$$B_2 = T A_2 T^{-1}.$$

The harmonic oscillator

The harmonic oscillator

The generator B_2 of the harmonic oscillator semigroup $U_m(t) = TS_m(\cdot)T^{-1}$ on $L^2(\mathcal{S}_m)$ is given by

$$D(B_2) = \left\{ f \in \widetilde{H^2}(\mathcal{S}_m) : f_i(0) = f_j(0),\ \forall i, j,\ \sum_{i=1}^{m} f_i'(0) = 0 \right\}$$

$$(B_2 f)_i(x) = \frac{1}{2}(f_i''(x) - x^2 f_i(x) + f_i(x)),\ \text{ for } f = (f_i)_{1\le i\le m} \in D(B_2).$$

The harmonic oscillator

The generator B_2 of the harmonic oscillator semigroup $U_m(t) = TS_m(\cdot)T^{-1}$ on $L^2(\mathcal{S}_m)$ is given by

$$D(B_2) = \left\{ f \in \widetilde{H^2}(\mathcal{S}_m) : f_i(0) = f_j(0),\ \forall i,j,\ \sum_{i=1}^{m} f_i'(0) = 0 \right\}$$

$$(B_2 f)_i(x) = \frac{1}{2}(f_i''(x) - x^2 f_i(x) + f_i(x)),\ \text{ for } f = (f_i)_{1 \le i \le m} \in D(B_2).$$

Moreover,

$$\sigma(B_2) = \{-k : k \in \mathbb{N}_0\},$$

where all even eigenvalues have multiplicity 1, whereas all odd eigenvalues have multiplicity $m - 1$.

Many thanks and stay healthy