Mean field game system and the master equation associated with local and nonlocal diffusions in the whole space

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What are the mean field games?

The mean field games (MFGs) were introduced independently by Lasry and Lions, and Caines, Huang and Malhamé in mid 2000s.

Idea: approximate finite games by games with infinitely many identical players. Focus on one sample player, all others are treated as a whole. Players move according to SDE

$$dX_t = \alpha_t \, dt + dL_t,$$

where L_t is a Lévy process with generator $\mathcal{L}^*$ and α_t is a control function. The goal is to minimize the cost functional:

$$\mathbb{E} \int_t^T (L(X_s, \alpha_s) + F(X_s, m(s))) \, ds + G(X_T, m(T)),$$

where m is the distribution of other players, F, G are the running and the terminal cost respectively and L is a Lagrangian. The optimal control is given by $-D_p H(x, Du)$, where the *value function* u satisfies the Hamilton–Jacobi–Bellman equation

$$-\partial_t u - \mathcal{L} u + H(x, Du) = F(x, m(t))$$

with terminal condition $G(x, m(T))$, H being the corresponding Hamiltonian.

The mean field game system

$$\begin{cases} -\partial_t u - \mathcal{L}u + H(x, Du(t,x)) = F(x, m(t)) & \text{in } (t_0, T) \times \mathbb{R}^d, \\ \partial_t m - \mathcal{L}^* m - \operatorname{div}\left(m D_p H(x, Du(t,x))\right) = 0 & \text{in } (t_0, T) \times \mathbb{R}^d, \\ m(t_0) = m_0, \qquad u(T,x) = G(x, m(T)). \end{cases} \tag{MFG}$$

- Here $H = H(x,p)$ and $\mathcal{L}$ is a Lévy (constant coefficient) operator:

$$\mathcal{L}u(x) = \underbrace{B \cdot Du(x)}_{\text{drift}} + \underbrace{\operatorname{div}\left(A \cdot Du(x)\right)}_{\text{diffusion}} + \underbrace{\int_{\mathbb{R}^d} \left(u(x+z) - u(x) - Du(x) \cdot z \mathbf{1}_{B(0,1)}(z)\right) \nu(dz)}_{\text{nonlocal diffusion/jumps}},$$

  where $b \in \mathbb{R}^d$, A is a positive semi-definite matrix, and $\int (1 \wedge |x|^2)\, \nu(dx) < \infty$.

- $F(x, m(t))$, $G(x, m(T))$ (instead of $m(t,x)/m(T,x)$) – nonlocal/smoothing coupling.

- Quick examples: $H(x,p) = |p|^2$, $F(x,m)/G(x,m) = m * \rho(x)$.

The mean field game system

$$
\begin{cases}
-\partial_t u - \mathcal{L} u + H(x, Du(t,x)) = F(x, m(t)) & \text{in } (t_0, T) \times \mathbb{R}^d, \\
\partial_t m - \mathcal{L}^* m - \operatorname{div}\left(m D_p H(x, Du(t,x))\right) = 0 & \text{in } (t_0, T) \times \mathbb{R}^d, \\
m(t_0) = m_0, \qquad u(T,x) = G(x, m(T)).
\end{cases} \tag{MFG}
$$

- System (MFG) consists of a backward Hamilton–Jacobi (H–J) equation and a forward Fokker–Planck (F–P) equation.
- We say that (u, m) solves (MFG) if u is a classical solution of H–J and m is a distributional solution of F–P ($m_0 \in \mathcal{P}(\mathbb{R}^d)$).
- $\mathcal{L}/L_t$ is the idiosyncratic noise. We work without common noise.

The mean field game system

$$\begin{cases} -\partial_t u - \mathcal{L}u + H(x, Du(t,x)) = F(x, m(t)) & \text{in } (t_0, T) \times \mathbb{R}^d, \\ \partial_t m - \mathcal{L}^* m - \operatorname{div}\left(m D_p H(x, Du(t,x))\right) = 0 & \text{in } (t_0, T) \times \mathbb{R}^d, \\ m(t_0) = m_0, \qquad u(T,x) = G(x, m(T)). \end{cases} \tag{MFG}$$

A few references:

$\mathcal{L} = \Delta$

- P.-L. Lions' lectures at Collège de France (notes by P. Cardaliaguet).
- Lasry–Lions: *Jpn. J. Math.* (2007), *C. R. Acad. Sci. Paris* (2006) $\mathbb{R}^d \rightsquigarrow \mathbb{T}^d$ (torus).
- Books by Carmona–Delarue: probabilistic approach via FBSDEs.
- ...

$\mathcal{L} = (-\Delta)^{\alpha/2}$

- Cesaroni et al., *J. Math. Pures Appl.* (2019), stationary on $\mathbb{T}^d$, $\alpha \in (1,2)$.
- Cirant–Goffi, *SIAM J. Math. Anal.* (2019), time-dependent on $\mathbb{T}^d$, $\alpha \in (0,2)$.

$\mathcal{L}$ – more general Lévy operator

- **Ersland–Jakobsen**, *J. Differ. Equ.* (2021), time-dependent on $\mathbb{R}^d$, order $\alpha \in (1,2)$.

The master equation

Assume that (u, m) solves (MFG) and let

$$U(t_0, x, m_0) = u(t_0, x).$$

(How will u evolve if we put m_0 as the initial distribution of players?)

Theorem (Well-posedness of the master equation)

If $F, G, H, \mathcal{L}$ are regular, then U is the unique solution of the following master equation.

$$\begin{cases} \partial_t U(t,x,m) = & -\mathcal{L}_x U(t,x,m) + H(x, D_x U(t,x,m)) - F(x,m) \\ & + \int_{\mathbb{R}^d} D_y \frac{\delta U}{\delta m}(t,x,m,y) H_p(y, D_y U(t,y,m))\, m(dy) \\ & - \int_{\mathbb{R}^d} \mathcal{L}_y \frac{\delta U}{\delta m}(t,x,m,y)\, m(dy) \quad \text{in } (0,T) \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d), \\ U(T,x,m) = & G(x,m) \quad \text{in } \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d). \end{cases}$$

- **P. Cardaliaguet, F. Delarue, J.-M. Lasry, and P.-L. Lions (CDLL). The master equation and the convergence problem in mean field games (2019).**
- **Master equation $\implies$ convergence of finite games to the mean field games.**
- We are looking for classical solutions, in particular all the derivatives of U must exist in the classical sense.

Metric in the space of probability measures

Let $m, m' \in \mathcal{P}(\mathbb{R}^d)$.

Kantorovich–Rubinstein/Lévy–Prokhorov distance

$$d_0(m, m') = \sup_{\|\phi\|_{W^{1,\infty}(\mathbb{R}^d)} \leqslant 1} \left| \int_{\mathbb{R}^d} \phi(x)\,(m' - m)(dx) \right|.$$

- d_0 is a metric for the narrow convergence of measures (tested with $C_b(\mathbb{R}^d)$).
- $(\mathcal{P}(\mathbb{R}^d), d_0)$ is complete.
- Unlike most of the works on MFGs in the whole space, it does not require that m, m' have any moments (compare e.g. to 1-Wasserstein distance d_1 which is equivalent to weak convergence + convergence of first moments).

Derivative in the space of probability measures

Let $m, m' \in \mathcal{P}(\mathbb{R}^d)$.

Derivative in $\mathcal{P}(\mathbb{R}^d)$

We say that $V\colon \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is C^1 if there exists a mapping $\frac{\delta V}{\delta m}\colon \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ continuous in both variables such that

$$\lim_{h\to 0^+} \frac{V(m + h(m' - m)) - V(m)}{h} = \int_{\mathbb{R}^d} \frac{\delta V}{\delta m}(m, y)\,(m' - m)(dy).$$

- Similar to the Gateaux derivative, but the space is not linear.
- Another perspective: $\int \frac{\delta V}{\delta m}(m - m') = \langle \frac{\delta V}{\delta m}, m - m' \rangle$, so $\frac{\delta V}{\delta m}\colon \mathcal{P}(\mathbb{R}^d) \to$ „$(\mathcal{M}(\mathbb{R}^d))^*$".
- To ensure that $\frac{\delta V}{\delta m}$ is uniquely defined CDLL use the following normalization:

$$\int_{\mathbb{R}^d} \frac{\delta V}{\delta m}(m, y) m(dy) = 0, \quad m \in \mathcal{P}(\mathbb{R}^d).$$

  Note: if $V(m) = \rho * m(x)$ for fixed $x \in \mathbb{R}^d$, then $\frac{\delta V}{\delta m}(m, y) \neq \rho(x - y)$ with normalization (even though it satisfies the definition).
- Fundamental theorem of calculus holds:

$$V(m') - V(m) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta V}{\delta m}(\lambda m + (1 - \lambda)m', y)\,(m' - m)(dy)\,d\lambda.$$

MFG system and the master equation

Formally, well-posedness of the master equation is quite easy given that we have well-posedness of the MFG system.

Existence

$$(u, m) \text{ solves MFG} \xrightarrow[\substack{\text{use H–J + F–P} \\ + \text{ regularity of } \frac{\delta U}{\delta m}}]{\text{Compute difference quotients}} U \text{ solves ME}$$

Uniqueness

$$V \text{ solves ME} \xrightarrow{\text{Fixed point argument}} \exists(\widetilde{m}(t)) \text{ satisfying F–P with } v(t,x) = V(t,x,\widetilde{m}(t))$$

$$\xrightarrow{\text{Use ME}} (v, \widetilde{m}) \text{ satisfies MFG} \xrightarrow[\text{uniqueness}]{\text{MFG}} (v, \widetilde{m}) = (u, m)$$

- In a sense, the MFG system and the master equation are equivalent.
- Existence and regularity of $\frac{\delta U}{\delta m}$ is the most involved element of proof.

A few references on the master equation

Analytic approach — most relevant for us:

- Cardaliaguet–Delarue–Lasry–Lions, chapter 3. **Torus/periodic boundary conditions**.
- M. Ricciardi. The master equation in a **bounded domain with Neumann conditions**. *Comm. PDE* (2022).
- Di Persio–Garbelli–Ricciardi. The master equation in a bounded domain with absorption. *arXiv:2203.15583*. **Dirichlet boundary conditions**.
- Graber–Sircar. Master equation for Cournot mean field games of control with absorption. *arXiv:2111:07020*.

Stochastic approach including the common noise:

- CDLL, chapters 4-6.
- Mou–Zhang (2019, 2022), Gangbo–Mészáros–Mou–Zhang (2021), Cardaliaguet–Cirant–Poretta (2022).

All the results above are for local diffusions.

Our contribution to the well-posedness of the master equation:

- Nonlocal, local and mixed diffusions.
- Handling the whole space for probability measures without moment conditions, using analytic methods (new even for $\mathcal{L} = \Delta$).

Our setting

The assumptions on F, G, H are rather standard and similar to the ones of CDLL.
We adopt the following condition for $\mathcal{L}$ from Ersland and Jakobsen:

There is $\mathcal{K} > 0$ and $\underline{\alpha} \in (1, 2]$, such that the **heat kernels** K and K^* of $\mathcal{L}$ and $\mathcal{L}^*$ respectively are densities of probability measures, and for $\tilde{K} = K, K^*$, $p \in [1, \infty)$ and $\beta \geqslant 0$ we have

$$\|D^\beta \tilde{K}(t, \cdot)\|_{L^p(\mathbb{R}^d)} \leqslant \mathcal{K} t^{-\frac{1}{\underline{\alpha}}(|\beta| + (1 - \frac{1}{p})d)}. \qquad \textbf{(K)}$$

In particular: $\|D^\beta \tilde{K}(t, \cdot)\|_{L^1(\mathbb{R}^d)} \leqslant \mathcal{K} t^{-|\beta|/\underline{\alpha}}$, crucial for Duhamel's formula:

$$\begin{cases} \partial_t u - \mathcal{L}u = f \\ u(0) = u_0 \end{cases} \iff u(t, x) = (K(t) * u_0)(x) + \int_0^t \int_{\mathbb{R}^d} K(t - s, x - y) f(s, y)\, dy\, ds.$$

Examples:

- $\mathcal{L} = (-\Delta)^{\alpha/2}$ for $\alpha \in (1, 2]$,
- $\nu(z) \approx |z|^{-d-\alpha}$ for $|z| \leqslant 1$, $\alpha \in (1, 2)$,
- $\mathcal{L} = (\partial^2_{x_1 x_1})^{\alpha_1/2} + (\partial^2_{x_2 x_2})^{\alpha_2/2} + \ldots + (\partial^2_{x_d x_d})^{\alpha_d/2}$ for $\alpha_1, \alpha_2, \ldots, \alpha_d > 1$,
- $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$, where $\mathcal{L}_1$ satisfies (**K**) and $\mathcal{L}_2$ is any Lévy operator.

Some intermediate results

- Modification of the well-posedness result for the MFG system. Ersland and Jakobsen considered classical solutions to both H–J and F–P. We obtain distributional solutions of class $C([t_0, T], \mathcal{P}(\mathbb{R}^d))$ for F–P, which allows any probability measure m_0 in the MFG system.
- Stability for the MFG system with respect to t_0 (overlooked in the literature).
- Existence, uniqueness and Hölder regularity of classical solutions for the equation $\partial_t z - \mathcal{L}z - b(t,x)Dz = f(t,x)$. We gain $\underline{\alpha} - \varepsilon$ derivatives over f, but it seems that (**K**) might be too weak to gain $\underline{\alpha}$.
- Existence and uniqueness of L^1 mild/distributional solutions for the equation $\partial_t \rho - \mathcal{L}\rho - \operatorname{div}(\rho V_1(t,x)) - \operatorname{div}(V_2(t,x)) = 0$, and an L^1 compactness result.

The linear system

Recall that (u, m) is the solution to the MFG system.

$\dfrac{\delta U}{\delta m}$ is obtained as the solution to the linearized system:

$$\begin{cases} -\partial_t z - \mathcal{L} z + D_p H(x, Du) \cdot Dz = \langle \frac{\delta F}{\delta m}(x, m(t)), \rho(t) \rangle & \text{in } (t_0, T) \times \mathbb{R}^d, \\ \partial_t \rho - \mathcal{L}^* \rho - \operatorname{div}(\rho D_p H(x, Du)) - \operatorname{div}(m D^2_{pp} H(x, Du) Dz) = 0 & \text{in } (t_0, T) \times \mathbb{R}^d, \\ z(T, x) = \langle \frac{\delta G}{\delta m}(x, m(T)), \rho(T) \rangle, \quad \rho(t_0) = \rho_0. \end{cases}$$

We have

$$\frac{\delta U}{\delta m}(t_0, x, m_0, y) = z(t_0, x), \quad z \text{ – solution for } \rho_0 = \delta_y,$$
$$\partial_y^\alpha \frac{\delta U}{\delta m}(t_0, x, m_0, y) = z(t_0, x), \quad z \text{ – solution for } \rho_0 = \partial^\alpha \delta_y \in C_b^{-|\alpha|}(\mathbb{R}^d),$$

so we need to handle irregular solutions of the ρ-equation. This is problematic in terms of determining the least possible regularity for F, G, H.

Due to duality, irregularity in ρ needs to be compensated by higher regularity of z (and so, of the coefficients). As a byproduct, our result on the master equation requires $4 + \varepsilon$ derivatives on u in x, which seems quite excessive.

Thank you for your attention!