Hölder continuity up to the boundary for kinetic equations

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Outline

Introduction

The classical Kolmogorov equation
Kinetic equations with rough coefficients
Kinetic equations with boundary

Ideas in the proofs

Galilean invariance and scaling
Weak solutions
Improvement of oscillation estimates

Kolmogorov equation

$f = f(t, x, v)$

In 1934, Kolmogorov computed the fundamental solution of

$$\partial_t f + v \cdot \nabla_x f - \Delta_v f = 0.$$

It is the following

$$K(t, x, v) = c_d t^{-2d} \exp \left\{ -\frac{|v|^2}{4t} - 3 \frac{(x - tv/2)^2}{t^3} \right\}.$$

It is similar to the usual heat kernel. We observe that solutions to this equation are **smooth**.

Kinetic equations

$$f = f(t, x, v)$$

Kinetic equations have the form

$$\partial_t f + v \cdot \nabla_x f - Q(f, f) = 0.$$

Here, $Q(f, f)$ is a nonlinear diffusion operator with respect to the v-variable.

- Boltzmann equation $\to$ $Q(f, f)$ is an integral operator.
- Landau equation $\to$ $Q(f, f)$ is a classical 2nd order diffusion.

De Giorgi/Nash/Moser for kinetic equations

Theorem

Given a kinetic equation with rough coefficients

$$f_t + v \cdot \nabla_x f + \partial_{v_i} a_{ij}(t,x,v) \partial_{v_j} f = -b_i \partial_{v_i} f + G,$$

where $b_i, G \in L^\infty$ and a_{ij} is uniformly elliptic.
$\Longrightarrow$ *f is Hölder continuous, with an estimate.*

This result is useful to derive smoothness estimates for the Landau equation.

- A. Pascucci & S. Polidoro [2004], $\|f\|_{L^\infty(Q_{1/2})} \lesssim \|f\|_{L^2(Q_1)}$.
- WD Wang & LQ Zhang [2009], $\|f\|_{C^\alpha(Q_{1/2})} \lesssim \|f\|_{L^\infty(Q_1)}$.
- F. Golse, C. Imbert, C. Mouhot & A. Vasseur [2016].
- J. Guerand & C. Imbert [2021].
- J. Guerand & C. Mouhot [2021].

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$\implies$ *f is Hölder continuous, with an estimate.*

Open problem: Consider

$$f_t + v \cdot \nabla_x f + a_{ij}(t,x,v) \partial_{v_i v_j} f = 0.$$

Is there an a priori estimate in C^α similar to Krylov-Safonov theorem for parabolic equations?

De Giorgi meets kinetic equations with integral-diffusion

Theorem (Imbert, S., 2016)

$$f_t + v \cdot \nabla_x f - \int_{\mathbb{R}^d} (f' - f) K \, \mathrm{d}v' = G.$$

We assume mild ellipticity and cancellation assumptions for K, and $G \in L^\infty$.

$\Longrightarrow$ *f is Hölder continuous, with an estimate.*

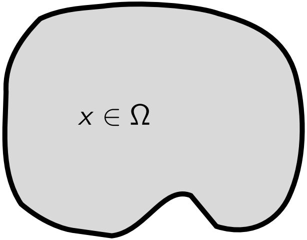
This result is useful to derive smoothness estimates for the Boltzmann equation.

(A simpler proof is given by Logan Stokols [2018] under stronger assumptions on the diffusion kernels)

A kinetic equation with boundary

Let us restrict x to a bounded domain Ω: $f : [0, T] \times \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}$.

$$f_t + v \cdot \nabla_x f + \partial_{v_i} a_{ij}(t, x, v) \partial_{v_j} f = -b_i \partial_{v_i} f + G,$$



with some boundary conditions for $x \in \partial\Omega$

There is a fresh preprint by Yuzhe Zhu about the same problem.

Here, x is restricted to a bounded domain and $v \in \mathbb{R}^d$. The opposite choice ($v \in \Omega$ and $x \in \mathbb{R}^d$), will be discussed in Litsgård's talk (I think).

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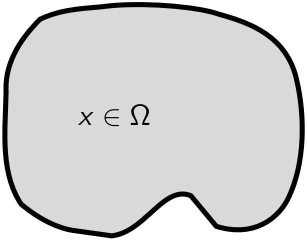

Specular reflection:
$f(t, x, v) = f(t, x, Rv)$
for $x \in \partial\Omega$, where
$Rv = v - 2(v \cdot n)n$.

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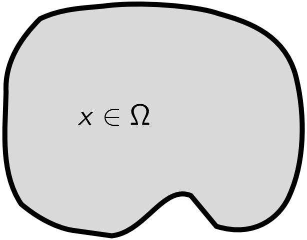

Influx: $f(t, x, v) = g(t, x, v)$,
on $\gamma_- = \{x \in \partial\Omega, v \cdot n < 0\}$.

No condition is given where $x \in \partial\Omega$ and $v \cdot n \geq 0$.

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A kinetic equation with boundary

Let us restrict x to a bounded domain Ω: $f : [0, T] \times \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}$.

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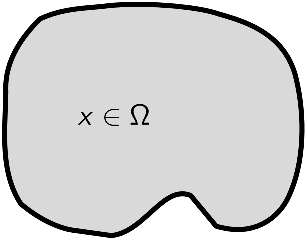

Other posibilities:
Diffuse boundary conditions,
bounce-back,
and lots of weird things...

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Specular reflection

Theorem

Given a kinetic equation with rough coefficients for $x \in \Omega$

$$f_t + v \cdot \nabla_x f + \partial_{v_i} a_{ij}(t, x, v) \partial_{v_j} f = -b_i \partial_{v_i} f + G,$$

where $b_i, G \in L^\infty$ and a_{ij} is uniformly elliptic.

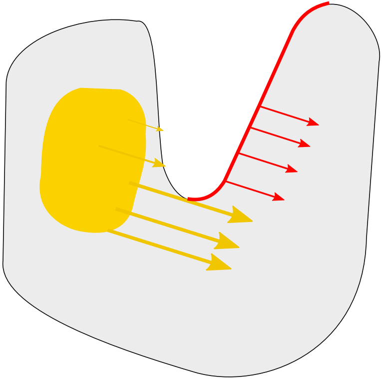
Consider the **specular reflection** *condition:*
$f(t, x, v) = f(t, x, Rv)$ *for $x \in \partial\Omega$.*
$\Longrightarrow$ *f is Hölder continuous up to the boundary with an estimate.*

The proof of this result follows from flattening the boundary and a mirror reflection explained in the paper *The Landau equation with the specular reflection boundary condition*, by Y. Guo, H.J. Hwang, J.W. Jang, Z. Ouyang, (2020).

Pure transport against the boundary

The transport equation with the influx condition generates discontinuities.

$$\partial_t f + v \cdot \nabla_x f = 0 \qquad \text{with } f = 0 \text{ on } x \in \partial\Omega, v \cdot n < 0.$$



Influx

Theorem (S. 2023)

For $x \in \Omega$,

$$f_t + v \cdot \nabla_x f + \partial_{v_i} a_{ij}(t,x,v) \partial_{v_j} f = -b_i \partial_{v_i} f + G,$$

where $b_i, G \in L^\infty$ and a_{ij} is uniformly elliptic.

With **influx** *condition: $f(t,x,v) = g$ for $x \in \partial\Omega$, $v \cdot n < 0$.*

For any $z_0 = (t_0, x_0, v_0) \in [0,T] \times \overline{\Omega} \times \mathbb{R}^d$,

$$\|f\|_{C^\alpha(Q_{1/2}(z_0) \cap \{x \in \Omega\})} \leq C \left(\|f\|_{L^2(Q_1(z_0) \cap \{x \in \Omega\})} + \|G\|_{L^\infty} + \|g\|_{C^\alpha} \right)$$

Moreover, if Ω is convex, the constants C and $\alpha > 0$ do not depend on z_0 or the shape of Ω.

When Ω is not convex, the constant C degenerates as $|v_0| \to \infty$.

The Upper bound

Theorem (S. 2023)

For $x \in \Omega$,

$$f_t + v \cdot \nabla_x f + \partial_{v_i} a_{ij}(t,x,v) \partial_{v_j} f \leq -b_i \partial_{v_i} f + G,$$

where $b_i, G \in L^\infty$ *and* a_{ij} *is uniformly elliptic.*

With **zero influx** *condition:* $f(t,x,v) = 0$ *for* $x \in \partial\Omega$, $v \cdot n < 0$.

For any $z_0 = (t_0, x_0, v_0) \in [0,T] \times \overline{\Omega} \times \mathbb{R}^d$,

$$\|f\|_{L^\infty(Q_{1/2}(z_0) \cap \{x \in \Omega\})} \leq C \left(\|f\|_{L^2(Q_1(z_0) \cap \{x \in \Omega\})} + \|G\|_{L^\infty} \right)$$

The constant C *does not depend on* z_0 *or the shape of* Ω *(even if not convex).*

Vanishing of infinite order on the incoming boundary

Theorem (S. 2023)

For $x \in \Omega$,

$$f_t + v \cdot \nabla_x f + \partial_{v_i} a_{ij}(t,x,v) \partial_{v_j} f = -b_i \partial_{v_i} f,$$

With **zero influx** *condition:* $f(t,x,v) = 0$ *for* $x \in \partial\Omega$, $v \cdot n < 0$.

For any $z_0 = (t_0, x_0, v_0) \in [0,T] \times \partial\Omega \times \mathbb{R}^d$, *so that* $v_0 \cdot n < 0$, *and* **any** $k > 0$,

$$|f(z)| \leq C_k \|f\|_{L^2(Q_1(z_0) \cap \{x \in \Omega\})} d(z, z_0)^k.$$

- If Ω is convex, the constants C_k do not deteriorate as $v_0 \to \infty$.
- If there is a nonzero boundary condition $f = g \in C^\alpha$, k is at most α.
- If there is a bounded source term $G \neq 0$, k is at most 2.

Inertial frames and scaling

Let us consider the Lie group structure

$$(t_1, x_1, v_1) \circ (t_2, x_2, v_2) := (t_1 + t_2, x_1 + x_2 + t_2 v_1, v_1 + v_2).$$

And the scaling

$$S_r(t, x, v) = (r^2 t, r^3 x, rv).$$

If f solves the Kolmogorov equation, so does

$$\tilde{f}(z) = f(z_0 \circ S_r z),$$

for any $z_0 \in \mathbb{R}^{1+2d}$ and $r > 0$.

Everything has to be in terms of these operations: convolutions, mollifiers, distance, cylinders, Hölder spaces, Sobolev spaces, etc...

Kinetic cylinders and distance

We define the unit kinetic cylinder Q_1 as

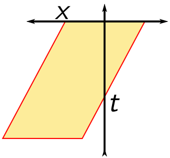
$$Q_1 := (-1, 0] \times B_1 \times B_1.$$

In order to recenter it at $z_0 \in \mathbb{R}^{1+2d}$ and rescale it, we use the invariances of the equation.

$$Q_r(z_0) := z_0 \circ S_r(Q_1).$$

We get an *oblique* cylinder in $\mathbb{R}^{1+2d}$,

$$Q_r(t_0, x_0, v_0) = \{(t, x, v) : \quad t - t_0 \in (-r^2, 0], \quad x - x_0 - (t - t_0)v_0 \in B_{r^3}, \quad v - v_0 \in B_r\}.$$



Also $d(z_1, z_2) \approx \inf\{r > 0 : z_1, z_2 \in Q_r(z) \text{ for some } z \in \mathbb{R}^d\}$.

Weak solutions

A weak solution of

$$f_t + v \cdot \nabla_x f + \partial_{v_i} a_{ij}(t,x,v)\partial_{v_j} f = 0,$$

$f = g$ in γ_-, is a function f so that

- $f \in L^2(D)$ and $\nabla_v f \in L^2(D)$
- for any smooth test function φ so that $\varphi = 0$ on γ_+, we have

$$\iiint_D -f(\partial_t + v \cdot \nabla_x)\varphi + a_{ij}\partial_{v_j} f \partial_{v_i}\varphi \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t$$
$$= -\iiint_{\gamma_-} g\varphi(v \cdot n) \, \mathrm{d}v \, \mathrm{d}S(x) \, \mathrm{d}t.$$

Here
$\gamma_- := \{x \in \partial\Omega, v \cdot n < 0\}$,
$\gamma_+ := \{x \in \partial\Omega, v \cdot n > 0\}$

Observations about weak solutions

A weak solution of

$$f_t + v \cdot \nabla_x f + \partial_{v_i} a_{ij}(t,x,v) \partial_{v_j} f = 0,$$

$f = g$ in γ_- by definition belongs to $L^2_{t,x} H^1_v$.

- From the equation $(\partial_t + v \cdot \nabla_x) f \in L^2_{t,x} H^{-1}_v$.
  (as in the kinetic Sobolev spaces in the paper by D. Albritton, S. Armstrong, J. -C. Mourrat, M. Novack)
- There is some notion of trace on $\partial\Omega$ in a weighted L^2 space.
- If $g = 0$, we can extend f as zero accross γ_-.
- There is no obvious way to extend f across γ_+
- If $\psi \in C^2$, the expected equation for $\psi(f)$ holds.

Weak sub-solutions

A **nonnegative** function f is a subsolution

$$f_t + v \cdot \nabla_x f + \partial_{v_i} a_{ij}(t,x,v) \partial_{v_j} f \leq 0,$$

$f = 0$ in γ_-, when its zero extension,

$$\tilde{f}(t,x,v) = \begin{cases} f(t,x,v) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

satisfies the inequality in the sense of distributions accross the whole boundary $\partial\Omega$.

Observations about nonnegative sub-solutions

$$f_t + v \cdot \nabla_x f + \partial_{v_i} a_{ij}(t,x,v) \partial_{v_j} f \leq 0, \text{ with } f = 0 \text{ on } \gamma_-.$$

- $(\partial_t + v \cdot \nabla_x) f$ may not be in $L^2_{t,x} H^{-1}_v$

  (it is **not** in the kinetic Sobolev space by D. Albritton, S. Armstrong, J. -C. Mourrat, M. Novack)
- It is unclear if there is any reasonable notion of trace on $\partial\Omega$.
- If f is a (sub)solution, ψ is convex, $\psi(0) = 0$, and $\psi(f) \geq 0$. Then $\psi(f)$ is a subsolution.
- If f is a solution with $f \leq 0$ on γ_-, f_+ is a nonnegative subsolution.

Upper bound up to the boundary

Theorem (Interior $L^2 \to L^\infty$)

$$f_t + v \cdot \nabla_x f + \partial_{v_i} a_{ij}(t,x,v)\partial_{v_j} f \leq -b_i \partial_{v_i} f + G, \text{ in } Q_1,$$

where $b_i, G \in L^\infty$ and a_{ij} is uniformly elliptic.

Then

$$\|f\|_{L^\infty(Q_{1/2})} \leq C \left(\|f\|_{L^2(Q_1)} + \|G\|_{L^\infty} \right).$$

This interior estimate implies the L^∞ estimate up to the boundary for subsolutions through the **extension as zero outside** Ω.

Main lemma for interior Hölder continuity

Lemma

Let $f : Q_1 \to [0,1]$ *be a subsolution*

$$f_t + v \cdot \nabla_x f + \partial_{v_i} a_{ij}(t,x,v) \partial_{v_j} f \leq -b_i \partial_{v_i} f + G, \text{ in } Q_1,$$

where $b_i, G \in L^\infty$ *and* a_{ij} *is uniformly elliptic. Assume*

$$|\{f = 0\} \cap Q^-| \geq \mu > 0.$$

Then $f \leq 1 - \theta$ *in* $Q_{1/2}$ *for some* $\theta > 0$ *depending on* μ*.*

- The zero-extension allows us to apply this lemma when we have $f = 0$ on γ_-.
- In a complete proof of Hölder continuity, we would want to apply this lemma to $(f - m)_+$ and $(m - f)_+$, but this would mess up the boundary condition on γ_-.
- When γ_- does not intersect $Q_1(z_0)$, there is no problem. Hölder continuity estimates are therefore easy if $\gamma_- \cap Q_1(z_0) = \emptyset$.

Convexity saves the day

Lemma

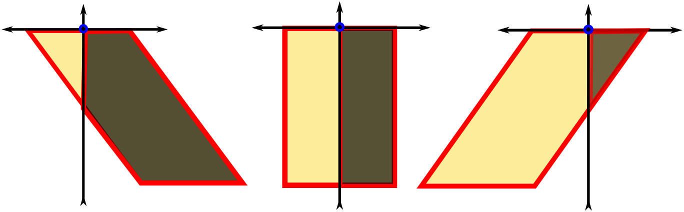
Assume that Ω **is convex**, *and* $z_1 \in \gamma_- \cap Q_{1/10}(z_0)$.
Then

$$|\{(t,x,v) \in Q^-(z_0) : x \notin \Omega\}| \geq \mu \text{ for some universal } \mu > 0.$$

- Note that $Q(z_0)$ and $Q^-(z_0)$ are oblique cylinders, with respect to v_0.
- Nothing like this would be true (uniformly in v_0) if convexity was replaced with an exterior ball condition.

Cylinders in different parts of the boundary

Oblique cylinders intersect the boundary differently depending on whether they are centered on γ_- or γ_+.



These are pictures of $Q_1(t,x,v)$ (only t and x), for $x \in \partial\Omega$.

left: $(t,x,v) \in \gamma_-$.
center: $(t,x,v) \in \gamma_0$.
right: $(t,x,v) \in \gamma_+$.

Iteration for Hölder continuity

Case 1. If $Q_{r/10}(z_0) \cap \gamma_- \neq \emptyset$:

Using the lemma: there is a big chunk of Q^- that is outside of Ω.

$$\|f\|_{L^\infty(Q_{r/2}(z_0)\cap\{x\in\Omega\})} \leq (1-\theta)\|f\|_{L^\infty(Q_r(z_0)\cap\{x\in\Omega\})}.$$

Case 2. If $Q_r(z_0) \cap \gamma_- = \emptyset$.

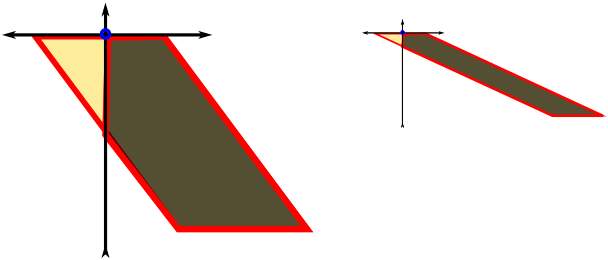
We proceed similarly as for the interior estimate and get

$$\operatorname*{osc}_{Q_{r/2}(z_0)\cap\{x\in\Omega\}} f \leq \operatorname*{osc}_{Q_r(z_0)\cap\{x\in\Omega\}} f.$$

Iterating this $\Rightarrow$ Hölder continuity.

Vanishing of infinite order

At small scales, the cylinders become more oblique.



If $z_0 \in \gamma_-$, most of $Q_r(z_0)$ is outside of Ω as $r \to 0$.

$$\Rightarrow \|f\|_{L^\infty(Q_{r/2}(z_0)\cap\{x\in\Omega\})} \leq \varepsilon_r \|f\|_{L^\infty(Q_r(z_0)\cap\{x\in\Omega\})},$$

with $\varepsilon_r \to 0$ as $r \to 0$.