# $L^p$ estimates for a class of degenerate operators

**Chiara Spina**

University of Salento

**Non-local operators, probability and singularities**

Cortona, 13-17 June 2022

Joint work with Giorgio Metafune and Luigi Negro.

We consider the operator

$$\mathcal{L} = \Delta_x + \Delta_y + c\frac{y}{|y|^2} \cdot \nabla_y - \frac{b}{|y|^2} = \Delta_x + L_y,$$

$b, \ c \in \mathbb{R}$, $x \in \mathbb{R}^N$, $y \in \mathbb{R}^M$ and $D := b + \left(\frac{M-2+c}{2}\right)^2 > 0$.

The operators $\Delta_x$, $L_y$ commute and the whole operator $\mathcal{L}$ satisfies the scaling property $I_s^{-1}\mathcal{L}I_s = s^2\mathcal{L}$, if $I_s u(x,y) = u(sx, sy)$.

Elliptic and parabolic solvability, generation of analytic semigroups in weighted spaces $L^p(\mathbb{R}^M, |y|^m dy)$, kernel estimates, maximal regularity, Rellich inequalities have been widely investigate for the operator $L_y$.

$$\mathcal{L} = \Delta_x + \Delta_y + c\frac{y}{|y|^2} \cdot \nabla_y - \frac{b}{|y|^2} = \Delta_x + L_y$$

We work in the space $L^p_c := L^p(\mathbb{R}^{N+M}, |y|^c\, dxdy)$, $d\mu = |y|^c\, dxdy$, under the assumption $M + c > 0$.

$\mathcal{L}$ generates an analytic semigroup in $L^p_c$ if and only if $L_y$ generates in $L^p(\mathbb{R}^M, |y|^c\, dy)$.

This happens if $(M + c)\left|\frac{1}{2} - \frac{1}{p}\right| < 1 + \sqrt{D}$.

**Remarks**

The weight $|y|^c$ makes the operator symmetric in $L^2_c$.

The assumption $M + c > 0$ insures that the measure $d\mu = |y|^c\, dx\, dy$ is locally finite on $\mathbb{R}^{N+M}$.

# Our main objectives

We investigate $L^p_c$-estimates for the pure $x$-derivatives of the form

$$\|D_{x_i x_j} u\|_p \leq C \|\mathcal{L} u\|_p.$$

The estimate

$$\|L_y u\| \leq C \|\mathcal{L} u\|_p$$

will immediately follow.

We will prove the boundedness of the operators $D_{x_i x_j} \mathcal{L}^{-1}$ or, in virtue of the scaling of $\mathcal{L}$, $D_{x_i x_j} (I - \mathcal{L})^{-1}$.

# Known results

When $M = 1$, that is in the half-space $\mathbb{R}^{N+1}_+$, $L^p$ estimates for $\mathcal{L}$ and other results have been proved (Metafune, Negro, Spina) by taking advantage of sophisticated tools from operator valued harmonic analysis.

More general, non symmetrizing weights $|y|^m \, dx \, dy$ are considered and both Dirichlet and Neumann boundary conditions.

# Known results

When $M = 1$, that is in the half-space $\mathbb{R}^{N+1}_+$, $L^p$ estimates for $\mathcal{L}$ and other results have been proved (Metafune, Negro, Spina) by taking advantage of sophisticated tools from operator valued harmonic analysis.

More general, non symmetrizing weights $|y|^m \, dx \, dy$ are considered and both Dirichlet and Neumann boundary conditions.

Similar results have also been obtained by H. Dong and T. Phan in the case $b = 0$ and for operators with variable coefficients in the second order part.

# The strategy

As **starting point** we get the claimed estimates in $L^2_c$. This is not hard and follows from difference quotients techniques.

The general $L^p$ result follows from sub-solution estimates through an **interpolation theorem in absence of kernels** in homogeneous spaces due to Z. Shen.

Sub-solution estimates, that is improving of integrability for (sub) solutions of the homogeneous equation $\mathcal{L}u = 0$, are proved by combining **Cacciopoli estimates**, **weighted Sobolev embeddings** and **Moser iteration**.

# The $L^2$ setting: some weighted spaces

$\Omega := \mathbb{R}^{N+M} \setminus \{(x,0),\ x \in \mathbb{R}^N\}$. $L^2_c = L^2(\mathbb{R}^{N+M}, |y|^c dxdy)$.

(i) $H^1_c(\Omega) = \{u \in H^1_{loc}(\Omega) :\ u,\ \nabla u \in L^2_c\}$ equipped with the norm

$$\|u\|_{H^1_c(\Omega)} := \|u\|_{L^2_c} + \|\nabla u\|_{L^2_c}.$$

(ii) $H^1_{0,c}(\Omega)$ defined as the closure of $C^\infty_c(\Omega)$ with respect to the norm of $H^1_c$.

(iii) $H^1_{c,\mathcal{D}}(\Omega) := \left\{ u \in H^1_c(\Omega)\ :\ \frac{u}{|y|} \in L^2_c \right\}$ equipped with the norm

$$\|u\|_{H^1_{c,\mathcal{D}}(\Omega)} := \|u\|_{L^2_c} + \|\nabla u\|_{L^2_c} + \left\| \frac{u}{|y|} \right\|_{L^2_c}.$$

# The operator in $L^2$

$\mathcal{L}$ can be put in divergence form:

$$\mathcal{L} = \Delta_x + \Delta_y + c\frac{y}{|y|^2} \cdot \nabla_y - \frac{b}{|y|^2} = |y|^{-c}\,\mathrm{div}\Big(|y|^c\,\nabla u\Big) - b|y|^{-2}.$$

We introduce the symmetric form $\mathfrak{a}$ in $L^2_c$

$$\mathfrak{a}(u,v) := \int_{\mathbb{R}^{N+M}} \left(\langle \nabla u, \nabla v\rangle + b|y|^{-2}u\bar{v}\right)\, d\mu,$$

$$u, v \in D(\mathfrak{a}) := H^1_{c,\mathcal{D}}(\Omega) = \left\{u \in H^1_c(\Omega) \ : \ \frac{u}{|y|} \in L^2_c\right\}.$$

$D(\mathfrak{a}) \subseteq H^1_{0,c}(\Omega)$.

$\mathcal{L}$ **generates** a contractive analytic semigroup $\left\{e^{z\mathcal{L}} : \ z \in \mathbb{C}_+\right\}$ in $L^2_c$ that extrapolates to $L^p_c$ for suitable $p$.

# $L^2$-estimates

By considering difference quotients, we get a regularity result for the $x$ and for the mixed derivatives in $L^2_c$.

Assume $D = b + \frac{(M-2+c)^2}{4} > 0$ and let $u \in D(\mathcal{L})$ be such that $u - \mathcal{L}u = f \in L^2_c$. Then for every $1 \le i,j \le N,\ 1 \le h \le M$, one has

$$\||y|^{-1}\nabla_x u\|_2 + \|D_{x_i y_h} u\|_2 + \|D_{x_i x_j} u\|_2 + \|L_y u\|_2 \le C\|f\|_2.$$

In other words, **the operator** $T = D_{x_i x_j}(I - \mathcal{L})^{-1}$ **is bounded in** $L^2_c$.

The $L^p_c$ estimates are more involved and to prove them we use a result due to Z. Shen in $(\mathbb{R}^{N+M}, d\mu)$.

# Shen's Theorem

Let $1 \leq p_0 < q_0 \leq \infty$. Suppose that $T$ is a sublinear bounded

operator on $L_c^{p_0}$.

Suppose moreover that there exist $\alpha_2 > \alpha_1 > 1$, $C > 0$ such that

$$\left(⨍_{Q_r} |Tf|^{q_0} d\mu\right)^{\frac{1}{q_0}} \leq C \left(⨍_{Q_{\alpha_1 r}} |Tf|^{p_0} d\mu\right)^{\frac{1}{p_0}}$$

for all cubes $Q_r$ of radius $r$ and for all $f \in C_c^\infty(\Omega)$, with support in $\Omega \setminus Q_{\alpha_2 r}$.

Then, for $p_0 \leq p < q_0$, there exists a positive constant $C_p$ such that for all $f \in C_c^\infty(\Omega)$

$$\|Tf\|_{L_c^p} \leq C_p \|f\|_{L_c^p}.$$

Notation:

$$⨍_Q f \, d\mu := \frac{1}{\mu(Q)} \int_Q f(y) \, d\mu, \qquad \mu(Q) = \int_Q |y|^c \, dy.$$

# How to apply Shen's Theorem

- $T = D_{x_i x_j}(I - \mathcal{L})^{-1}$ is bounded on $L^2_c$.

# How to apply Shen's Theorem

- $T = D_{x_i x_j}(I - \mathcal{L})^{-1}$ is bounded on $L^2_c$.
- Let $q_0 > 2$ be in the generation range. Let $Q_r$ be a cube in $\Omega$ and $f \in C_c^\infty(\Omega)$ with support in $\mathbb{R}^{N+M} \setminus Q_{4r}$.

## How to apply Shen's Theorem

- $T = D_{x_i x_j}(I - \mathcal{L})^{-1}$ is bounded on $L^2_c$.
- Let $q_0 > 2$ be in the generation range. Let $Q_r$ be a cube in $\Omega$ and $f \in C_c^\infty(\Omega)$ with support in $\mathbb{R}^{N+M} \setminus Q_{4r}$.
- Set $v = Tf = D_{x_i x_j}(I - \mathcal{L})^{-1} f$. Since $f = 0$ in $Q_{4r}$ and $D_{x_i x_j}(I - \mathcal{L})^{-1} = (I - \mathcal{L})^{-1} D_{x_i x_j}$,

$$v - \mathcal{L}v = D_{x_i x_j} f = 0 \quad \text{in} \quad Q_{4r}.$$

## How to apply Shen's Theorem

- $T = D_{x_i x_j}(I - \mathcal{L})^{-1}$ is bounded on $L^2_c$.
- Let $q_0 > 2$ be in the generation range. Let $Q_r$ be a cube in $\Omega$ and $f \in C_c^\infty(\Omega)$ with support in $\mathbb{R}^{N+M} \setminus Q_{4r}$.
- Set $v = Tf = D_{x_i x_j}(I - \mathcal{L})^{-1} f$. Since $f = 0$ in $Q_{4r}$ and $D_{x_i x_j}(I - \mathcal{L})^{-1} = (I - \mathcal{L})^{-1} D_{x_i x_j}$,

$$v - \mathcal{L} v = D_{x_i x_j} f = 0 \quad \text{in} \quad Q_{4r}.$$

- In order to apply Shen's Theorem, we should prove

$$\left( \fint_{Q_r} |v|^{q_0} d\mu \right)^{\frac{1}{q_0}} \leq C \left( \fint_{Q_{2r}} |v|^2 d\mu \right)^{\frac{1}{2}}.$$

# How to apply Shen's Theorem

- $T = D_{x_i x_j}(I - \mathcal{L})^{-1}$ is bounded on $L^2_c$.
- Let $q_0 > 2$ be in the generation range. Let $Q_r$ be a cube in $\Omega$ and $f \in C_c^\infty(\Omega)$ with support in $\mathbb{R}^{N+M} \setminus Q_{4r}$.
- Set $v = Tf = D_{x_i x_j}(I - \mathcal{L})^{-1} f$. Since $f = 0$ in $Q_{4r}$ and $D_{x_i x_j}(I - \mathcal{L})^{-1} = (I - \mathcal{L})^{-1} D_{x_i x_j}$,

$$v - \mathcal{L}v = D_{x_i x_j} f = 0 \quad \text{in} \quad Q_{4r}.$$

- In order to apply Shen's Theorem, we should prove

$$\left( \fint_{Q_r} |v|^{q_0} d\mu \right)^{\frac{1}{q_0}} \leq C \left( \fint_{Q_{2r}} |v|^2 d\mu \right)^{\frac{1}{2}}.$$

- It will follow that $T$ extends to a bounded operator in $L^p_c$ for every $2 < p < q_0$.

# How to apply Shen's Theorem

- $T = D_{x_i x_j}(I - \mathcal{L})^{-1}$ is bounded on $L^2_c$.
- Let $q_0 > 2$ be in the generation range. Let $Q_r$ be a cube in $\Omega$ and $f \in C_c^\infty(\Omega)$ with support in $\mathbb{R}^{N+M} \setminus Q_{4r}$.
- Set $v = Tf = D_{x_i x_j}(I - \mathcal{L})^{-1} f$. Since $f = 0$ in $Q_{4r}$ and $D_{x_i x_j}(I - \mathcal{L})^{-1} = (I - \mathcal{L})^{-1} D_{x_i x_j}$,

$$v - \mathcal{L}v = D_{x_i x_j} f = 0 \quad \text{in} \quad Q_{4r}.$$

- In order to apply Shen's Theorem, we should prove

$$\left( \fint_{Q_r} |v|^{q_0} d\mu \right)^{\frac{1}{q_0}} \leq C \left( \fint_{Q_{2r}} |v|^2 d\mu \right)^{\frac{1}{2}}.$$

- It will follow that $T$ extends to a bounded operator in $L^p_c$ for every $2 < p < q_0$.
- The boundedness for $p < 2$ follows from the self-adjointness of $T = D_{x_i x_j}(I - \mathcal{L})^{-1}$.

# Shen's assumption and Sobolev inequalites

We should prove that,

$$\left(\fint_{Q_r} |v|^{q_0} d\mu\right)^{\frac{1}{q_0}} \leq C \left(\fint_{Q_{2r}} |v|^2 d\mu\right)^{\frac{1}{2}}$$

for solutions $v$ of $v - \mathcal{L}v = 0$ in $Q_{4r}$ and for $q_0 \geq 2$ when $b \geq 0$ and $2 \leq q_0 < \frac{M+c}{s_1}$ when $b < 0$ and $M - 2 + c > 0$. Here $s_1 = \frac{M-2+c}{2} - \sqrt{D}$.

# Shen's assumption and Sobolev inequalites

We should prove that,

$$\left(\fint_{Q_r} |v|^{q_0} d\mu\right)^{\frac{1}{q_0}} \leq C \left(\fint_{Q_{2r}} |v|^2 d\mu\right)^{\frac{1}{2}}$$

for solutions $v$ of $v - \mathcal{L}v = 0$ in $Q_{4r}$ and for $q_0 \geq 2$ when $b \geq 0$ and $2 \leq q_0 < \frac{M+c}{s_1}$ when $b < 0$ and $M - 2 + c > 0$. Here $s_1 = \frac{M-2+c}{2} - \sqrt{D}$.

From weighted local Sobolev inequalities, for $c \geq 0$, $N + M \geq 2$,

$$\left(\fint_{Q_R} |u|^q \, d\mu\right)^{\frac{1}{q}} \leq CR \left(\fint_{Q_R} |\nabla u|^2 \, d\mu\right)^{\frac{1}{2}}$$

for functions $u \in H^1_{0,c}(\Omega)$ (and in particular in $D(\mathfrak{a})$) with $\operatorname{supp} u \subseteq Q_R$, for $2 \leq q < 2^*_c$ with $\frac{1}{2^*_c} = \frac{1}{2} - \frac{1}{N+M+c}$.

# Caccioppoli inequalities

Let $D = b + \frac{(M-2+c)^2}{4} > 0$, $0 < r < R$. Let $u \in D(a)$ be sub-solution of $u - \mathcal{L}u = 0$ in $Q_R$ and $s \geq 2$ such that $D - b^- \frac{(s-2)^2}{4(s-1)} > 0$.

If $(u^+)^{\frac{s}{2}} \in L^2_c(Q_R)$ then $\nabla (u^+)^{\frac{s}{2}} \in L^2_c(Q_r)$ and

$$(s-1)\int_{Q_r} |\nabla u|^2 (u^+)^{s-2}\, d\mu + b^+ \int_{Q_r} \frac{(u^+)^s}{|y|^2}\, d\mu \\ \leq \frac{C(b,c,M)}{(R-r)^2} \int_{Q_R} (u^+)^s\, d\mu.$$

The assumption $D - b^- \frac{(s-2)^2}{4(s-1)} > 0$ allows to apply some Hardy's type inequalities. When $b \geq 0$, Caccioppoli inequalities hold for every $s \geq 2$.

# Caccioppoli inequalities with Sobolev embeddings

Let $D > 0$, $N + M \geq 2$, $2 \leq q < 2_c^*$. Then there exists $C = C(q, N, M, b, c) > 0$ such that

$$\left( \fint_{Q_r} |u^+|^{\frac{q}{2}s} \, d\mu \right)^{\frac{2}{qs}} \leq$$

$$\left( \frac{R}{r} \right)^{\frac{2(N+M+c^+)}{qs}} R^{\frac{2}{s}} \left( \frac{Cs}{(R-r)^2} \right)^{\frac{1}{s}} \left( \fint_{Q_R} |u^+|^s \, d\mu \right)^{\frac{1}{s}}$$

for $u \in D(a)$ sub-solution of $u - \mathcal{L}u = 0$ in $Q_R$ s.t. $u^+ \in L_c^s(Q_R)$ for $s \geq 2$ with $D - b^- \frac{(s-2)^2}{4(s-1)} > 0$.

We iterate this procedure until when $D - b^- \frac{(s-2)^2}{4(s-1)} > 0$. When $b < 0$ we should stop after a finite number of steps. This depends on the validity of some Hardy inequalities.

# Moser's iteration

Let $D > 0$, $b < 0$, $M + c - 2 > 0$, $c \geq 0$, $0 < \alpha < 1$. If $u \in D(a)$ is a sub-solution of the equation $u - \mathcal{L}u = 0$ in $Q_r$, for any $2 \leq q_0 < \kappa \frac{M+c}{s_1}$ there exists $C(\alpha, q_0, M, b, c) > 0$ such that

$$\left( \fint_{Q_{\alpha r}} (u^+)^{q_0} \, d\mu \right)^{\frac{1}{q_0}} \leq C \left( \fint_{Q_r} |u|^2 \, d\mu \right)^{\frac{1}{2}} .$$

Here $\kappa = \frac{N+M+c}{N+M+c-2} \frac{M+c-2}{M+c} < 1$ (if $N \neq 0$).

For $N = 0$, $\kappa = 1$ and we cover the generation range!

# Remarks

When $b \geq 0$, we use Moser's iteration procedure to get sup-norm estimates.

When $b < 0$ and $M - 2 + c > 0$, Moser's iteration allows to prove mean value inequalities for values of $q_0$ strictly smaller than $\kappa \frac{M+c}{s_1}$ for some $\kappa < 1$ unless $N = 0$.

This happens because Moser's iteration can be performed only with a finite number of steps until Hardy inequality can be applied.

When $c \geq 0$ an additional sectional argument combined with the known results for $L_y$ allows to reach the optimal exponent $q_0 = (M + c)/s_1$.

# Optimal mean value inequalities

Let $q_{max} = \frac{M+c}{s_1}$. Set $\overline{q} = \sup A$ where $A$ is the set of $2 \leq q < q_{max}$ for which Shen's assumption is satisfied.

If $2 \leq q < q_{max}$, $q \in A$, then

$$\left( \fint_{Q_a} |u|^q d\mu \right)^{\frac{1}{q}} \leq C(q,a,b) \left( \fint_{Q_b} |u|^2 d\mu \right)^{\frac{1}{2}}$$

for every $a < b < 1$ and $u \in D(\mathcal{L})$ s.t. $u - \mathcal{L}u = f$, $f = 0$ in $Q$. $(2 \in A)$.

We will prove that $\overline{q} = q_{max}$. Assume $\overline{q} < q_{max}$ and let $q \in A$.

Let $u \in D(\mathcal{L})$ such that $u - \mathcal{L}u = 0$ in $Q$. Then it is easy to prove that $\left( \fint_{Q_a} |\Delta_x u|^q d\mu \right)^{\frac{1}{q}} < \infty$.

The equation $u - \mathcal{L}u = 0$ in $Q$ can be written

$$u - \Delta_y u - c\frac{y}{|y|^2}\nabla_y u + \frac{b}{|y|^2}u = \Delta_x u.$$

$Q = Q^N \times Q^M$ ($Q^N \in \mathbb{R}^N$, $Q^M \in \mathbb{R}^M$).

For fixed $x \in Q^N$, consider the operator $L_y = \Delta_y + c\frac{y}{|y|^2}\nabla_y - \frac{b}{|y|^2}$ and the equation

$$u - \Delta_y u - c\frac{y}{|y|^2}\nabla_y u + \frac{b}{|y|^2}u = g \qquad \text{in } \mathbb{R}^M$$

where $g \in L^q_c(\mathbb{R}^M)$, $g = \Delta_x u$ in $Q^M_a$ and $g = 0$ outside.

- The existence results for $L_y$ in $L^q_c(\mathbb{R}^M)$,
- the domain embeddings for $L_y$,
- the optimality of the improved inequality via Moser when $N = 0$ (that is in $\mathbb{R}^M$),
- classical Sobolev embedding Theorems

give

$$\left( \fint_{Q_a} |u|^r \, d\mu \right)^{\frac{1}{r}} \leq C \left( \fint_{Q_b} |u|^2 \, d\mu \right)^{\frac{1}{2}}$$

for every $a < b < 1$ and for some $\overline{q} < r < q_{max}$. Then $r \in A$, in contrast with $r > \overline{q} = \sup A$.

# Optimal mean value inequalities

Assume $N+M\geq 2$, $b<0$, $c\geq 0$ such that $D>0$, $M-2+c>0$.

If $0<\alpha<1$ and $2\leq q_0<\frac{M+c}{s_1}$ then there exists a positive constant $C$ such that, if $u\in D(\mathcal{L})$ satisfies $u-\mathcal{L}u=f$ in $Q_r$,

$$\left(\fint_{Q_{\alpha r}}|u|^{q_0}d\mu\right)^{\frac{1}{q_0}}\leq C\left(\fint_{Q_r}|u|^2d\mu\right)^{\frac{1}{2}}.$$

# $L^p$ estimates

Let $D = b + \frac{(M-2+c)^2}{4} > 0$. Assume that

$$(i)\ M = 1 \text{ and } |c| < 1, \quad \text{or} \quad (ii)\ b \geq 0, \quad \text{or} \quad (iii)\ b < 0 \text{ and } c \geq 0.$$

Then, for every $1 \leq i, j \leq N$, the operators $D_{x_i x_j}(I - \mathcal{L})^{-1}$, originally defined in $L^2_c$, extend to bounded operators in $L^p_c$ for every $p$ satisfying $(M + c)\left|\frac{1}{2} - \frac{1}{p}\right| < 1 + \sqrt{D}$.