Regularity results for classical solutions to degenerate Kolmogorov equations

Bianca Stroffolini

Università di Napoli Federico II

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Outline

- Degenerate Kolmogorov equations;
- our main result;
- blow-up method;
- Taylor polynomial.

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Degenerate Kolmogorov Equations

Prototype equations I

$$\partial_t f(v,x,t) + \langle v, \nabla_x f(v,x,t)\rangle = Q(f(v,x,t)) \qquad (v,x,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \tag{1}$$

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$Q(f)$ is the "collision operator"

- if $Q(f)=\mathrm{div_v}(\nabla_\mathrm{v}\mathrm{f}+\mathrm{vf})$,(1) becomes the linear Fokker-Planck equation;
- in the Boltzmann-Landau equation $Q(f)=\sum_{i,j=1}^n \partial_{v_i}(a_{ij}(\cdot,f)\partial_{v_j}f)$.

Prototype equations II

$$S^2 \partial_{SS} V + f(S) \partial_M V - \partial_t V = 0 \qquad S, t > 0, M \in \mathbb{R} \qquad (2)$$

with either $f(S) = \log S$ or $f(S) = S$ arises in the Black and Scholes theory.

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Heath-Jarrow-Morton model

$$df(t,s) = \mu(t,s)dt + \sum_i \sigma_{\mathrm{i}}(t,s) dW_{i\,t}$$

where W_t is a d-dimensional Wiener process and μ and σ are adapted processes.

Fundamental solution

[Kolmogorov](1934)

$$\Gamma(v,x,t) = \frac{\sqrt{3^n}}{(2\pi t^2)^n} \exp\left(-\frac{|v|^2}{t} - 3\frac{\langle v,x\rangle}{t^2} - 3\frac{|x|^2}{t^3}\right)$$

Bei einigen physikalisch ganz natürlichen Nebenvoraussetzungen folgt aus (2–8), dass G die Fundamentallösung der folgenden Differentialgleichung vom Focker-Planckschen Typus ist:[4]

$$\frac{\partial g}{\partial t'} = -\Sigma\, \dot q'_i \frac{\partial}{\partial q'_i} g - \Sigma \frac{\partial}{\partial \dot q'_i}\{f_i(t', q', \dot q')g\} + \Sigma\Sigma \frac{\partial^2}{\partial \dot q'_i\, \partial \dot q'_j}\{k(t', q', \dot q')g\}. \tag{9}$$

Im Falle $n = 1$ hat man also die Gleichung

$$\frac{\partial g}{\partial t'} = -\dot q' \frac{\partial}{\partial q'} g - \frac{\partial}{\partial \dot q'}\{f(t', q', \dot q')g\} + \frac{\partial^2}{\partial \dot q'^2}\{k(t', q', \dot q')g\}. \tag{10}$$

Wenn f und k konstant sind, so findet man als die fundamentale Lösung von (10) den Ausdruck

$$g = \frac{2\sqrt{3}}{\pi k^2 (t'-t)^2} \exp\left\{-\frac{[\dot q' - \dot q - f(t'-t)]^2}{4k\,(t'-t)} - \frac{3\left[q' - q - \frac{\dot q' + \dot q}{2}(t'-t)\right]^2}{k^3\,(t'-t)^3}\right\}. \tag{11}$$

Define for $(v, x, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+$

$$f(v,x,t) = \int_{\mathbb{R}^n} \Gamma(v - \upsilon, x - \xi + t\upsilon, t) f_0(\upsilon, \xi) d\upsilon d\xi +$$
$$\int_{\mathbb{R}^n \times]0,t[} \Gamma(v - \upsilon, x - \xi + (\tau - t)\upsilon, t - \tau) g(\upsilon, \xi, \tau) d\upsilon d\xi d\tau$$

Then f is a solution to the Cauchy problem

$$\begin{cases} \partial_t f(v,x,t) + \langle v, \nabla_x f(v,x,t) \rangle = \Delta_v f(v,x,t) + g(v,x,t) \\ f(v,x,0) = f_0(v,t) \end{cases}$$

Invariance properties

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- Translation: $\widetilde{f}(v,x,t) := f(v+\upsilon, x+\xi+\upsilon t, t+\tau)$

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Note that

$$f(v,x,t) = \int_{\mathbb{R}^2\times]0,t[} \Gamma((\upsilon,\xi,\tau)^{-1}\circ(v,x,t))\, g(\upsilon,\xi,\tau) d\upsilon\, d\xi\, d\tau.$$

Hypoellipticity

Theorem ([Hörmander] - 1967)

Let f be a (distributional) solution to $X_1^2 f + \cdots + X_m^2 f + Yf = g$ in $\Omega \subset \mathbb{R}^N \times \mathbb{R}$. If

$$span\Big\{ Y, X_1, \dots, X_m, [X_i, X_j], [X_l, Y], \dots, [X_i, \dots, [X_j, X_l]] \Big\} = \mathbb{R}^{N+1}$$

Then

$$g \in C^\infty(\Omega) \quad \Rightarrow \quad f \in C^\infty(\Omega).$$

Commutators: $[X_i, X_j] f := X_i X_j f - X_j X_i f$

Kolmogorov operator

$$\mathscr{L} := \partial_v^2 - v\partial_x - \partial_t = X^2 + Y$$

- $X = \partial_v, \quad Y = -v\partial_x - \partial_t,$

$$Y \sim -\begin{pmatrix}0\\v\\1\end{pmatrix} \qquad X \sim \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad [X,Y] = XY - YX \sim -\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$[X,-Y]f := \partial_v(v\partial_x + \partial_t)f - (v\partial_x + \partial_t)\partial_v f = \partial_x f$$

Geometry

$$\mathscr{L} := \partial_v^2 - v\partial_x - \partial_t = X^2 + Y$$

$$X = \partial_v \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad Y = -v\partial_x - \partial_t \sim - \begin{pmatrix} 0 \\ v \\ 1 \end{pmatrix} \quad [X, Y] = -\partial_x \sim - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$(0,0,0) \xrightarrow{\delta X} (\delta, 0, 0) \xrightarrow{\eta Y} (\delta, -\delta\eta, -\eta) \xrightarrow{-\delta X} (0, -\delta\eta, -\eta) \xrightarrow{-\eta Y} (0, -\delta\eta, 0)$$

More general operators

$$\mathscr{L} u := \sum_{j,k=1}^{m} a_{jk} \partial^2_{y_j y_k} u + \sum_{j,k=1}^{N} b_{jk} y_k \partial_{y_j} u - \partial_t u$$

With $A = (a_{jk})_{j,k=1,\dots,m}$, $B = (b_{jk})_{j,k=1,\dots,N}$ constant coefficients matrices, $0 < m \le N$,

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$A = A(y,t) = (a_{jk}(y,t))_{j,k=1,\dots,m}$ with Dini continuous coefficients.

Assumption on B

There exists a basis of $\mathbb{R}^N$ such that

$$B = \begin{pmatrix} * & * & \dots & * & * \\ B_1 & * & \dots & * & * \\ \mathbb{O} & B_2 & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & B_\kappa & * \end{pmatrix}$$

Every B_j is a $m_j \times m_{j-1}$ matrix of rank m_j, with

$$m_0 \geq m_1 \geq \ldots \geq m_\kappa \geq 1, \quad \text{and} \quad m_0 + m_1 + \ldots + m_\kappa = N.$$

Notation

$$E(t) := \exp(-tB)$$

- Invariance group
  $(x, t) \circ (\xi, \tau) = (\xi + E(\tau)x, t + \tau), \quad (x, t), (\xi, \tau) \in \mathbb{R}^{N+1}$
- Dilation $\delta_r := \mathrm{diag}(rI_m, r^3 I_{m_1}, \dots, r^{2\kappa+1} I_{m_\kappa}, r^2)$
- Homogeneous norm:
  $\|(x, t)\|_{\mathbb{K}} := \max\left\{|x_1|^{\frac{1}{\alpha_1}}, \dots, |x_N|^{\frac{1}{\alpha_N}}, |t|^{\frac{1}{2}}\right\}$ where
  $\alpha_1, \dots, \alpha_{m_0} = 1, \alpha_{m_0+1}, \dots, \alpha_{m_0+m_1} = 3, \alpha_{N-m_\kappa}, \dots, \alpha_N = 2\kappa + 1$
- Distance: $d_{\mathbb{K}}((x, t), (\xi, \tau)) :=$
  $\|(\xi, \tau)^{-1} \circ (x, t)\|_{\mathbb{K}}, \qquad (x, t), (\xi, \tau) \in \mathbb{R}^{N+1}.$

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- Distance: $d_{\mathbb{K}}((x,t),(\xi,\tau)) := \|(\xi,\tau)^{-1} \circ (x,t)\|_{\mathbb{K}}, \qquad (x,t),(\xi,\tau) \in \mathbb{R}^{N+1}.$

$$\mathcal{Q}_r(x_0,t_0) := \{(x,t) \in \mathbb{R}^{N+1} : d_{\mathbb{K}}((x,t),(x_0,t_0)) < r\}$$

Classical solutions

- We define Yu as a Lie Derivative:

$$Yu(x,t) := \lim_{s\to 0} \frac{u(\exp(sB)x, t-s) - u(x,t)}{s}.$$

- We say that u belongs to $C^2_{\mathscr{L}}(\Omega)$ if $u, \partial_{v_i} u, \partial_{v_i v_j} u$ $(i,j = 1,\ldots,m)$ and Yu are continuous function.

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- Moreover we require that

$$\lim_{s\to 0} \frac{\partial_{x_i} u(\exp(sB)x, t-s) - \partial_{x_i} u(x,t)}{|s|^{1/2}} = 0,$$

uniformly on compact subsets of Ω, for every $i = 1,\dots,m$.

The homogeneous operator $\mathscr{L}_0$

If every block denoted by $*$ has zero entries, the associated operator will be denoted by $\mathscr{L}_0$. This is the only case where the operator $\mathscr{L}$ is invariant with respect to the family of dilations δ_r.

The operator $\mathscr{L}_0$ is the blow-up limit of the operator $\mathscr{L}$.

We define $\mathscr{L}_r$ as the *scaled operator of* $\mathscr{L}$ in terms of $(\delta_r)_{r>0}$ as follows

$$\mathscr{L}_r := r^2(\delta_r \circ \mathscr{L} \circ \delta_{\frac{1}{r}}), \tag{3}$$

$$\mathscr{L}_r = \sum_{i,j=1}^{m} a_{ij}\partial^2_{x_i x_j} + Y_r, \quad r \in (0,1] \tag{4}$$

where

$$Y_r := \langle B_r x, D\rangle - \partial_t \tag{5}$$

and $B_r := r^2\delta_r B\delta_{\frac{1}{r}}$, i.e.,

$$B_r = \begin{pmatrix} r^2 B_{0,0} & r^4 B_{0,1} & \dots & r^{2\kappa} B_{0,\kappa-1} & r^{2\kappa+2} B_{0,\kappa} \\ B_1 & r^2 B_{1,1} & \dots & r^{2\kappa-2} B_{1,\kappa-1} & r^{2\kappa} B_{1,\kappa} \\ \mathbb{O} & B_2 & \dots & r^{2\kappa-4} B_{2,\kappa-1} & r^{2\kappa-2} B_{2,\kappa} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & B_\kappa & r^2 B_{\kappa,\kappa} \end{pmatrix}. \qquad (6)$$

Clearly, $\mathscr{L}_r = \mathscr{L}$ for every $r > 0$ if and only if $B = B_0$, and the principal part operator $\mathscr{L}_0$ is obtained as the limit of (3) as $r \to 0$. Setting $E_r(t) = \exp(-tB_r)$, we define the translation group related to $\mathscr{L}_r$ as

$$(x, t) \circ_r (\xi, \tau) = (\xi + E_r(\tau)x, t + \tau), \quad (x, t), (\xi, \tau) \in \mathbb{R}^{N+1}. \quad (7)$$

Note that the composition law defined in (7) depends continuously on $r \in (0, 1]$.

Known results

Schauder estimates for Hölder continuous coefficients of A

- [Manfredini] (1997)
- [Lunardi] (1997)
- [Lorenzi] (2000)(2005)
- [Di Francesco, Polidoro] (2006)

Main results

Dini continuous functions

- The continuity modulus of g on $\Omega \subset \mathbb{R}^{2n+1}$ is

$$\omega_g(r) := \sup_{\substack{(v,x,t),(\upsilon,\xi,\tau)\in\Omega \\ d((v,x,t),(\upsilon,\xi,\tau))<r}} |g(v,x,t) - g(\upsilon,\xi,\tau)|.$$

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  We say that g is Dini-continuous on Ω if

$$\int_0^1 \frac{\omega_g(r)}{r} dr < +\infty.$$

Main result

Let $u \in C^2_{\mathscr{L}}(\mathcal{Q}_1(0,0,0))$ be a classical solution to $\mathscr{L} u = g$. Suppose that g is Dini continuous. Then

$$|\partial^2 u(v,x,t) - \partial^2 u(\upsilon,\xi,\tau)| \leq c\Big(d \sup_{\mathcal{Q}_1(0,0,0)} |u| + d \sup_{\mathcal{Q}_1(0,0,0)} |g| + \int_0^d \frac{\omega_g(r)}{r} + d \int_d^1 \frac{\omega_g(r)}{r^2} \Big),$$

for every (v,x,t) and $(\upsilon,\xi,\tau) \in \mathcal{Q}_{\frac{1}{2}}(0,0,0)$.

Here $d = d((v,x,t),(\upsilon,\xi,\tau))$ and $\partial^2 u$ denotes either $\partial^2_{v_i v_j} u$, for $i,j = 1,\ldots,m$, or Yu.

Main result (cont)

Moreover

$$|\partial^2 u(0,0,0)| \leq c \left(\sup_{\mathcal{Q}_1(0,0,0)} |u| + |g(0,0,0)| + \int_0^1 \frac{\omega_g(r)}{r} \right).$$

An analogous result holds true when the coefficients of the matrix A are Dini continuos.

Corollary

- Hölder continuous function $\omega_g(r) = r^\alpha, 0 < \alpha < 1$.

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  Then $\partial^2 u$ is α-Hölder continuous (Schauder estimates)
- Lipschitz continuous function $\omega_g(r) = r$, then

$$|\partial^2 u(v,x,t) - \partial^2 u(\upsilon,\xi,\tau)| \leq c\, d \sup_{\mathcal{Q}_1(0,0,0)} |u| + c\, d |\log d| \, \|g\|_{C_L^{0,1}(\mathcal{Q}_1(0,0,0))}.$$

About the proof

Method introduced in [Wang](2006)

- Blow-up
  - weak maximum principle
  - mean value formulas
- Taylor polynomial

Our approach

- Blow-up
  - a priori interior estimates based on representation formulas
  - maximum principle
- Taylor polynomial

Blow-up

Method of the proof

Denote $\mathcal{Q}_k := \mathcal{Q}_{\varrho_k}(0,0,0)$ with $\varrho_k = \frac{1}{2^k}$,

and consider the sequence of Dirichlet problems

$$\begin{cases} \mathscr{L} u_k = g(0,0,0), & \text{in } \mathcal{Q}_k \\ u_k = u, & \text{in } \partial\mathcal{Q}_k \end{cases}$$

We need to estimate

$$|\partial^2 u(z) - \partial^2 u(0)| \leq |\partial^2 u(z) - \partial^2 u_k(z)| + |\partial^2 u_k(z) - \partial^2 u_k(0)| + |\partial^2 u_k(0) - \partial^2 u(0)|$$

for every $z = (v, x, t) \in \mathcal{Q}_2$.

Convergence

$$\begin{cases} \mathscr{L} u_k = g(0,0,0), & \text{in } \mathcal{Q}_k \\ u_k = u, & \text{in } \partial \mathcal{Q}_k \end{cases}$$

$$|\partial^2 u(z) - \partial^2 u(0)| \le |\partial^2 u_k(z) - \partial^2 u_k(0)| + |\partial^2 u_k(0) - \partial^2 u(0)| + |\partial^2 u(z) - \partial^2 u_k(z)|$$

Main tools:

- maximum principle;
- interior a priori estimates;
- Taylor expansion for u.

Continuity

$$\begin{cases} \mathscr{L} u_k = g(0,0,0), & \text{in } \mathcal{Q}_k \\ u_k = u, & \text{in } \partial\mathcal{Q}_k \end{cases}$$

$$|\partial^2 u(z) - \partial^2 u(0)| \leq |\partial^2 u_k(z) - \partial^2 u_k(0)| + |\partial^2 u_k(0) - \partial^2 u(0)| + |\partial^2 u(z) - \partial^2 u_k(z)|$$

Main tools:

- A priori interior estimates;
- Lagrange theorem.

Estimate of $|\partial^2 u_k(0) - \partial^2 u(0)|$

Recall that $\mathscr{L} u_k = g$. The function $v_k := u - u_k$ is a solution to

$$\begin{cases} \mathscr{L} v_k = g - g(0), & \text{in } \mathcal{Q}_k \\ v_k = 0, & \text{in } \partial\mathcal{Q}_k \end{cases}$$

The maximum principle implies

$$\|u_k - u\|_\infty \le 4\varrho^{2k}\|g - g(0)\|_\infty \le 4\varrho^{2k}\omega_g(\varrho^k).$$

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Moreover $\mathscr{L}(u_k - u_{k+1}) = 0$ in $\mathcal{Q}_{k+1}$, then $i, j = 1, \dots, m$

$$\begin{aligned} \|\partial^2(u_k - u_{k+1})\|_{L^\infty(\mathcal{Q}_{k+2})} &\le C\varrho^{-2k-4} \sup_{\mathcal{Q}_{k+1}} |u_k - u_{k+1}| \\ &\le C\varrho^{-2k}\varrho^{2k}\omega_g(\varrho^k) = C\omega_g(\varrho^k). \end{aligned}$$

Conclusion

Let $k \geq 1$ be such that $\varrho^{k+4} \leq d(z,0) \leq \varrho^{k+3}$. Then

$$\boxed{\sum_{j=k}^{\infty} |\partial^2 u_j(0) - \partial^2 u_{j+1}(0)| \leq C \sum_{j=k}^{\infty} \omega_g(\varrho^j) \leq C \int_0^{d(z,0)} \frac{\omega_g(r)}{r} dr.}$$

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Remark - The Taylor expansion of u is needed to prove that

$$\sum_{j=k}^{\infty} \left(\partial^2 u_j(0) - \partial^2 u_{j+1}(0)\right) \quad \text{converges to} \quad \partial^2 u_k(0) - \partial^2 u(0).$$

Taylor strikes back I

To this end, we first consider the derivative $\partial^2_{x_i x_j} u_k$ and we prove that

$$\lim_{k \to +\infty} \partial^2_{x_i x_j} u_k(0) = \partial^2_{x_i x_j} T^2_0 u(0), \tag{8}$$

where $T^2_0 u(\zeta)$ is the second-order Taylor polynomial of u around the origin, computed at some point $\zeta = (\xi, \tau) \in \mathcal{Q}_k$: Thus, by applying Taylor Theorem to $u \in C^2_{\mathscr{L}}(\mathcal{Q}_1(0))$, we obtain that

$$\lim_{k \to +\infty} \partial^2_{x_i x_j} u_k(0) = \partial^2_{x_i x_j} u(0). \tag{9}$$

Taylor strikes back II

We compute $\mathscr{L} T_0^2 u$ in $\zeta = (\xi, \tau)$ as

$$\mathscr{L} T_0^2 u(\zeta) = \sum_{i,j=1}^{m} a_{ij} \partial^2_{\xi_i \zeta_j} u(0) - \partial_t u(0) + \langle v, \xi \rangle + \langle M\xi, \xi \rangle$$

where v is a constant vector of $\mathbb{R}^N$ and M is a $N \times N$ constant matrix.

In addition, as $\mathscr{L} u = f$ in $\mathcal{Q}_k$, we have that

$$\sum_{i,j=1}^{m} a_{ij} \partial^2_{\xi_i \zeta_j} u(0) - \partial_t u(0) = \mathscr{L}_0 u(0) = \mathscr{L} u(0) = f(0) \qquad (10)$$

and thus

$$\mathscr{L} T_0^2 u(\zeta) = f(0) + \langle v, \xi \rangle + \langle M\xi, \xi \rangle. \qquad (11)$$

Thus, the definition of u_k gives us

$$\mathscr{L}\left(T_0^2 u - u_k\right)(\zeta) = \langle v, \xi\rangle + \langle M\xi, \xi\rangle, \quad \zeta \in \mathcal{Q}_k. \tag{12}$$

We now apply a-priori estimates to $T_0^2 u - u_k$ for $R = \varrho^k$ and infer

$$|\partial^2_{x_i x_j}(u_k - T_0^2 u)(0)| \le C\varrho^{-2k} \sup_{\mathcal{Q}_k} |u_k - T_0^2 u| + O(\varrho^k). \tag{13}$$

Moreover, $T_0^2 u$ is the second-order Taylor polynomial of u:

$$\sup_{\zeta \in \mathcal{Q}_k} |u - T_0^2 u| = o(\varrho^{2k}) \tag{14}$$

Thus, from estimates (14) and estimates on v_k, we obtain

$$\sup_{\mathcal{Q}_k} |u_k - T_0^2 u| \le \sup_{\mathcal{Q}_k} |v_k| + \sup_{\mathcal{Q}_k} |u - T_0^2 u| \le o(\varrho^{2k}). \tag{15}$$

Estimates (13) and (15) finally yield

$$|\partial^2_{x_i x_j}(u_k - T_0^2 u)(0)| \le C\varrho^{-2k} o(\varrho^{2k}) + O(\varrho^k) \le o(1),$$

Estimate of $|\partial^2 u_k(z) - \partial^2 u_k(0)|$

Set $w_k := u_k - u_{k+1}$ ($\varrho^{k+4} \le d(z,0) \le \varrho^{k+3}$) and write

$u_k(z) - u_k(0) = u_0(z) - u_0(0) + \sum_{j=0}^{k-1} (w_j(0) - w_j(z))$

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Use the Lagrange theorem and the interior a priori estimates

$$|\partial^2 u_k(z) - \partial^2 u_k(0)| \le C d(z,0) + \left(\|u_0\|_{L^\infty(\mathcal{Q}_0)} \sum_{j=0}^{k-1} \varrho^{-j} \omega_g(\varrho^j) \right) \le$$

$$C d(z,0) \left(\|u_0\|_{L^\infty(\mathcal{Q}_0)} + \|g\|_{L^\infty(\mathcal{Q}_1(0))} + \int_{d(z,0)}^1 \frac{\omega_g(r)}{r^2} \right)$$

Conclusion

We have proved that, if g is Dini continuous, then

$$|\partial^2 u(v,x,t) - \partial^2 u(0,0,0)| \leq c\Big(d(z,0) \sup_{\mathcal{Q}_1(0,0,0)} |u| + d(z,0) \sup_{\mathcal{Q}_1(0,0,0)} |g| + \int_0^{d(z,0)} \frac{\omega_g(r)}{r} + d(z,0) \int_{d(z,0)}^1 \frac{\omega_g(r)}{r^2} \Big),$$

for every $(v,x,t) \in \mathcal{Q}_2(0,0,0)$.

Taylor polynomial

Taylor polynomial

If $u \in C^2_{\mathscr{L}}(\Omega)$ and $z \in \Omega$, then

$$T^2_z u(\zeta) := u(z) + \sum_{i=1}^{m} \partial_{v_i} u(z)(\upsilon_i - v_i) + \frac{1}{2} \sum_{i,j=1}^{m} \partial^2_{v_i v_j} u(z)(\upsilon_i - v_i)(\upsilon_j - v_j) - Yu(z)(\tau - t)$$

is its second order Taylor polynomial at z.

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Theorem [PRS] If $u \in C^2_{\mathscr{L}}(\Omega)$ and $z \in \Omega$, then

$$u(\zeta) - T^2_z u(\zeta) = o(d(z,\zeta)^2) \quad \text{as} \quad \zeta \to z.$$

Comparison with literature

- Folland and Stein, using a quantitative version of the Carathéodory-Chow-Rashevsky connectivity result and a Mean Value Theorem.
- [Bonfiglioli](2008);
- [Pagliarani, Pascucci, Pignotti](2015).

They consider the case $n = 1, \alpha = 1$
In the above articles the derivatives are assumed to be Hölder continuous.

The method of the proof relies on the construction of a finite sequence of points which connect $z = (x, t)$ and $\zeta = (\xi, \tau)$ and are located along suitable trajectories. More precisely, we start from z and choose $z_1 = (x_1, t_1)$ as the point along the integral curve of the drift Y satisfying the condition $t_1 = \tau$. We then move along the integral paths of $X_1, \ldots, X_m$ to a point $z_2 = (x_2, t_2)$ such that $x_2^{[0]} = \xi^{[0]}$ and $t_2 = \tau$. This allows us to exploit the regularity of u along the vector fields $X_1, \ldots, X_m, Y$ and estimate the remainder in terms of the homogeneous norm of the new points.

We use an induction argument: As a preliminary result, we prove our claim under the assumption that the points $z = (x, t)$ and $\zeta = (\xi, \tau)$ have the same temporal component $t = \tau$.

Base case $n = 0$. In this case, we are only changing the variables x_i, for $i = 1, \dots, m$, moving along the direction $e^{s_0 X_{v_0}}$ where $v_0 = (v_{0,1}, \dots, v_{0,m}, 0 \dots, 0)$ is a suitable unit vector in V_0.

$$T_z^2 u(\zeta) = u(x,t) + \sum_{i=1}^{m} \partial_{x_i} u(x,t) s_0 v_{0,i} + \frac{s_0^2}{2} \sum_{i,j=1}^{m} \partial^2_{x_i,x_j} u(x,t) v_{0,i} v_{0,j}. \quad (16)$$

We observe that $\|z^{-1} \circ \zeta\|_K^2 = |s_0|^2$ and therefore we want to show that

$$u(\zeta) - T_z^2 u(\zeta) = o(|s_0|^2) \quad \text{as} \quad s_0 \to 0. \quad (17)$$

Use the multidimensional euclidean mean-value theorem used the continuity of the second order derivatives of u.

Sketch of the proof

$$z_0 = (0,0,0) \xrightarrow{sX} z_1 = (s,0,0) \xrightarrow{s^2 Y} z_2 = (s, -s^3. - s^2)$$

$$z_2 \xrightarrow{-sX} z_3 = (0, -s^3, -s^2) \xrightarrow{-s^2 Y} (0, -s^3, 0) = z_4$$

Sketch of the proof

$$\begin{aligned} u(z_4) - T^2_{z_0} u(z_4) = & \boxed{u(z_4) - u(z_3) - s^2 Yu(z_3)} \\ & + \boxed{u(z_3) - T^2_{z_2} u(z_3)} \\ & - \boxed{u(z_1) - u(z_2) - s^2 Yu(z_2)} \\ & + \boxed{T^2_{z_1} u(z_0) - u(z_0)} \\ & + \boxed{T^2_{z_2} u(z_3) - u(z_2) - T^2_{z_1} u(z_0) + u(z_1)} \\ & + \boxed{s^2 (Yu(z_3) - Yu(z_2))} - \boxed{T^2_{z_0} u(z_4) - u(z_0)} \end{aligned}$$

By the inductive hypothesis, the second and the forth difference are $o(|s|^2)$ as $s \to 0$.
We now observe that

$$u(z_4) - u(z_3) = u(e^{-s^2 Y}(z_3)) - u(z_3).$$

By applying the mean value theorem along the direction of the drift, we find that there exists $\bar{s}$ such that

$$u(e^{-s^2 Y}(z_3)) - u(z_3) = -s^2 Yu(e^{\bar{s}Y}(z_3)),$$

where $|\bar{s}| \leq |s|$. Similarly we obtain that

$$u(z_2) - u(z_1) = s^2 Yu(e^{\widetilde{s}Y}(z_1)),$$

where again $\widetilde{s}$ verifies $|\widetilde{s}| \leq |s|$.
By letting $s \to 0$, we find that $\bar{s}, \widetilde{s} \to 0$, and therefore, using the continuity of Yu, we have showed that the sum of the first and the third difference is again equal to $o(|s|^2)$ as $s \to 0$.

The 5th term

$$\boxed{\left| T^2_{z_2} u(z_3) - u(z_2) - T^2_{z_1} u(z_0) + u(z_1) \right|} \le$$

$$s \sum_{i=1}^{m} |\partial_{v_i} u(z_2) - \partial_{v_i} u(z_1)| +$$

$$\frac{s^2}{2} \sum_{i,j=1}^{m} |\partial^2_{v_i v_j} u(z_2) - \partial^2_{v_i v_j} u(z_1)| + s^2 |Yu(z_2) - Yu(z_1)|$$

Here we use the assumption

$$\lim_{\tau \to 0} \frac{\partial_{v_i} u(v, x - \tau v, t - \tau) - \partial_{v_i} u(v, x, t)}{|\tau|^{1/2}} = 0$$

Many thanks for your attention!