Relativistic stable operators with critical potentials

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We want to solve

$$\partial_t u = \mathcal{L}u$$

where

$$\mathcal{L}f(x) := \int_{\mathbb{R}^d} (f(x+z) - f(x) - \mathbb{1}_{|z|<1} \langle z, \nabla f(x) \rangle) \kappa(x,z) J(z)\, dz$$

$$= \int_{\mathbb{R}^d} \left(f(x+z) - f(x) - \mathbb{1}_{|z|<r} \langle z, \nabla f(x) \rangle \right) \frac{\kappa(x,z)}{|z|^{d+\alpha}} dz$$
$$+ \underbrace{\left(\int_{\mathbb{R}^d} z \left(\mathbb{1}_{|z|<r} - \mathbb{1}_{|z|<1} \right) \frac{\kappa(x,z)}{|z|^{d+\alpha}} dz \right)}_{internal\ \ drift} \cdot \nabla f(x) \,.$$

Results: existence, uniqueness, estimates, regularity, semigroup analysis
Technique: parametrix method

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Example

Let $J(z)$ be comparable to

$$\nu(|z|) = |z|^{-d-1}\varphi(|z|)\,,$$

$$\varphi(r) := \begin{cases} c_k\, r^{-1/4}\,, & r \in \left[((2k+1)!)^{-1}, ((2k)!)^{-1}\right], \\ c_k \sqrt{(2k+1)!}\, r^{1/4}\,, & r \in \left[((2k+2)!)^{-1}, ((2k+1)!)^{-1}\right], \end{cases}$$

and $c_k = ((2k)!!)^{-1/2}$. We put $\varphi(r) = 0$ if $r > 1$.

Then

$$\int_{|z|<1} |z|\,\nu(|z|)dz = \infty\,.$$

WLSC(α) holds with $\alpha = 3/4$, but fails for any $\alpha > 3/4$
WUSC(β) holds with $\beta = 5/4$, but fails for any $\beta < 5/4$.

Literature

Based on:

[4] J. Minecki, K. Szczypkowski
Non-symmetric Lévy-type operators

[3] K. Szczypkowski
Regularity of fundamental solutions for Lévy-type operators

[2] K. Szczypkowski
Fundamental solution for super-critical non-symmetric Lévy-type operators

[1] T. Grzywny, K. Szczypkowski
Heat kernels of non-symmetric Lévy-type operators

[0] T. Grzywny, K. Szczypkowski
Estimates of heat kernels of non-symmetric Lévy processes

Bogdan, Böttcher, Chen, Kim, Knopova, Kochubei, Kühn, Kulczycki, Kulik, Menozzi, Ryznar, Schilling, Song, Sztonyk, Vondracek, Zhang, ...

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Literature

Based on:

[1] T. Jakubowski, K. Kaleta, K. Szczypkowski
Relativistic stable operator with critical potential

[BGJP] Bogdan, Grzywny, Jakubowski, Pilarczyk

[BHJ] Bogdan, Hansen, Jakubowski

[BBS] Bogdan, Butko, Szczypkowski

[BJS] Bogdan, Jakubowksi, Sydor

[Roncal] Luz Roncal

Let $V\colon \mathbb{R}^d \to [0,\infty]$. We consider

$$Lf(x) + V(x)f(x)\,.$$

Examples: for $\alpha \in (0,2)$, $\kappa > 0$, $m \geq 0$,

a) $-(-\Delta)^{\alpha/2} + \kappa|x|^{-\alpha}$

b) $-(-\Delta + m^{2/\alpha})^{\alpha/2} + \kappa|x|^{-\alpha}$ (appl. in physics $d=3$, $\alpha=1$)

c) $-(-\Delta + m^{2/\alpha})^{\alpha/2} + V(x)$

Aim: establish estimates of the corresponding transition density (heat kernel)

$$p \sim -(-\Delta)^{\alpha/2} \qquad \tilde{p} \sim -(-\Delta)^{\alpha/2} + \kappa|x|^{-\alpha}$$

$$p \sim -(-\Delta + m^{2/\alpha})^{\alpha/2} \qquad \tilde{p} \sim -(-\Delta + m^{2/\alpha})^{\alpha/2} + V(x)$$

Schrödinger perturbations

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Schrödinger perturbations of transition densities

A function $p\colon (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d\to[0,\infty]$ is a **transition density** if

$$\int_{\mathbb{R}^d} p(s,x,z)p(t-s,z,y)\,dz = p(t,x,y)\,,\qquad t>0,\ x,y\in\mathbb{R}^d\,.$$

For $V\colon\mathbb{R}^d\to[0,\infty]$ we define **the Schrödinger perturbation of** p **by** V as

$$\tilde{p}_V=\sum_{n=0}^{\infty}p_n\,,$$

where $p_0(t,x,y)=p(t,x,y)$ and for $n\geq 1$,

$$p_n(t,x,y)=\int_0^t\int_{\mathbb{R}^d}p_{n-1}(s,x,z)V(z)p(t-s,z,y)\,dzds\,.$$

Remark: The function $\tilde{p}_V$ is a transition density [BHJ] and

$$\tilde{p}_V(t,x,y)=p(t,x,y)+\int_0^t\int_{\mathbb{R}^d}\tilde{p}_V(s,x,z)V(z)p(t-s,z,y)\,dzds\,.$$

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Remark: The function $\tilde{p}_V$ is a transition density [BHJ] and

$$\tilde{p}_V(t,x,y) = p(t,x,y) + \int_0^t\int_{\mathbb{R}^d} \tilde{p}_V(s,x,z)V(z)p(t-s,z,y)\,dzds\,.$$

Schrödinger perturbations of transition densities

If

$$\int_s^\infty \int_{\mathbb{R}^d} p(u-s,x,z)\left[\partial_u \phi(u,z) + L_z \phi(u,z)\right] dzdu = -\phi(s,x)\,,$$

then ("modulo absolute integrability") by [BJS] we have

$$\int_s^\infty \int_{\mathbb{R}^d} \tilde{p}_V(u-s,x,z)\left[\partial_u \phi(u,z) + L_z \phi(u,z) + V(z)\phi(u,z)\right] dzdu = -\phi(s,x)\,.$$

The hypothesis holds for almost any semigroup [BBS].

Fact 1. Suppose that for some $\varepsilon \in [0,1)$ and all $t \in (0,1]$, $x, y \in \mathbb{R}^d$,

$$p_1(t,x,y) \le \varepsilon p(t,x,y)$$

then

$$p(t,x,y) \le \tilde{p}(t,x,y) \le (1-\varepsilon)^{-1} p(t,x,y)\,.$$

Proof. We get $p_n(t,x,y) \le \varepsilon^n p(t,x,y)$. □

How to guarantee the hypothesis? By 3G inequality

$$p(s,x,z)p(t-s,z,y) \le c p(t,x,y)\Big[p(s,x,z) + p(t-s,z,y)\Big]$$

and

$$\int_0^t \int_{\mathbb{R}^d} p(s,x,z)V(z)\,dzds \le \varepsilon/2 \qquad \dots\,.$$

Schrödinger perturbation by Kato potentials

Fact 1. Suppose that for some $\varepsilon \in [0,1)$ and all $t \in (0,1]$, $x, y \in \mathbb{R}^d$,

$$p_1(t,x,y) = \int_0^t \int_{\mathbb{R}^d} p(s,x,z)V(z)p(t-s,z,y)\,dzds \leq \varepsilon p(t,x,y)$$

then

$$p(t,x,y) \leq \tilde{p}(t,x,y) \leq (1-\varepsilon)^{-1} p(t,x,y)\,.$$

Proof. We get $p_n(t,x,y) \leq \varepsilon^n p(t,x,y)$. □

How to guarantee the hypothesis? By 3G inequality

$$p(s,x,z)p(t-s,z,y) \leq cp(t,x,y)\Big[p(s,x,z) + p(t-s,z,y)\Big]$$

and

$$\int_0^t \int_{\mathbb{R}^d} p(s,x,z)V(z)\,dzds \leq \varepsilon/2 \qquad \ldots\,.$$

Transition densities: α-stable and relativistic α-stable

For $m \geq 0$ let

$$p^m(t,x,y) = \int_0^\infty g_s(x-y) e^{-m^{2/\alpha} s} \eta_t(s)\, ds\,,$$

where $g_s(x)$ is G-W kernel and $\eta_t(s)$ the kernel of $\alpha/2$-stable subordinator.

$$p^0 \sim -(-\Delta)^{\alpha/2} \qquad \text{and} \qquad p^1 \sim -(-\Delta+1)^{\alpha/2}\,.$$

Recall that

$$-(-\Delta)^{\alpha/2} f(x) = p.v. \int_{\mathbb{R}^d} \left(f(x+z) - f(x)\right) \frac{c_{d,\alpha}}{|z|^{d+\alpha}}\, dz$$

$$\left(-(-\Delta+1)^{\alpha/2} + 1\right) f(x) = p.v. \int_{\mathbb{R}^d} \left(f(x+z) - f(x)\right) \nu(|z|)\, dz$$

where

$$\nu(r) = c_{d,\alpha} r^{-\frac{d+\alpha}{2}} K_{\frac{d+\alpha}{2}}(r)\,.$$

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$$p^0 \sim -(-\Delta)^{\alpha/2} \qquad \text{and} \qquad p^1 \sim -(-\Delta+1)^{\alpha/2}\,.$$

We have

$$p^0(t,x,y) \approx t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \qquad \text{on} \quad (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d,$$

and

$$p^1(t,x,y) \approx t^{-d/\alpha} \wedge t\nu(|x-y|) \qquad \text{on} \quad (0,T] \times \mathbb{R}^d \times \mathbb{R}^d,$$

where

$$\nu(r) = c_{d,\alpha} r^{-\frac{d+\alpha}{2}} K_{\frac{d+\alpha}{2}}(r)\,.$$

Fact 2. The heat kernel p^0 corresponding $-(-\Delta)^{\alpha/2}$ satisfies 3G

$$p^0(s,x,z)p^0(t-s,z,y) \le cp^0(t,x,y)\Big[p^0(s,x,z) + p^0(t-s,z,y)\Big]\,.$$

Corollary 1. Let $V(x) = \kappa|x|^{-\alpha+\theta}$, $0 < \theta \le \alpha$. Then

$$\int_0^t \int_{\mathbb{R}^d} p^0(s,x,z)\,V(z)\,dzds \le ct^{\theta/\alpha}\,,$$

and for all $t \in (0,T]$, $x,y \in \mathbb{R}^d$

$$p^0(t,x,y) \le \tilde{p}^0(t,x,t) \le C(T)\,p^0(t,x,y)\,.$$

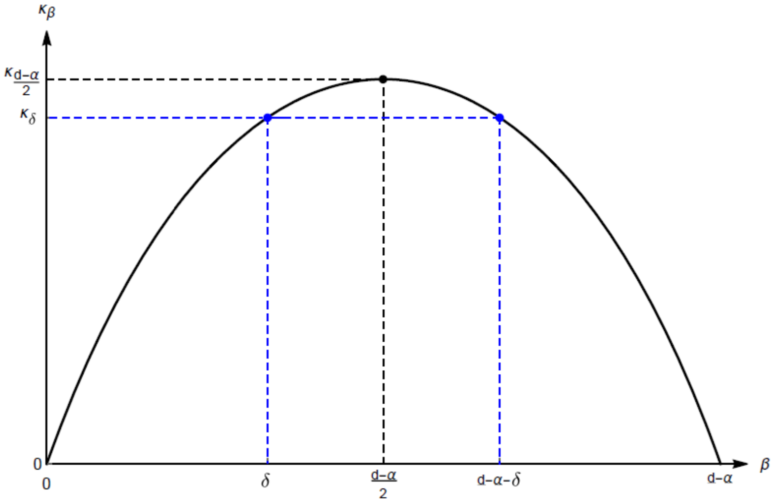
Criticality:

$$\lim_{|x|\to 0} \int_0^t \int_{\mathbb{R}^d} p^0(s,x,z)|z|^{-\alpha}\,dzds = \infty\,.$$

Critical potentials: α-stable and relativistic α-stable

In [BGJP]: for $\beta \in (0, d-\alpha)$ let

$$V^0_\beta(x) := \kappa_\beta |x|^{-\alpha}, \qquad \kappa_\beta = \frac{2^\alpha \Gamma\left(\frac{\beta+\alpha}{2}\right)\Gamma\left(\frac{d-\beta}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(\frac{d-\beta-\alpha}{2}\right)}.$$



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where

$$\nu(r) = c_{d,\alpha} r^{-\frac{d+\alpha}{2}} K_{\frac{d+\alpha}{2}}(r)\,.$$

Critical potentials: α-stable and relativistic α-stable

Consider ([BDK], [Roncal])

$$V_\beta^1(x) := \frac{2^{\alpha/2}\Gamma\left(\frac{d-\beta}{2}\right)}{\Gamma\left(\frac{d-\beta-\alpha}{2}\right)} \frac{K_{\frac{\beta+\alpha}{2}}(|x|)}{K_{\frac{\beta}{2}}(|x|)} |x|^{-\alpha/2} .$$

$V_\beta^0(x) = \kappa_\beta |x|^{-\alpha}$

Lemma 1. We have $V_\beta^1 - V_\beta^0 > 0$.

Lemma 2. We have

$$\lim_{|x|\to 0} \frac{V_\beta^1(x)}{V_\beta^0(x)} = 1 .$$

Lemma 3. For $|x| \le 1/2$,

$$V_\beta^1(x) - V_\beta^0(x) \approx \begin{cases} |x|^{2-\alpha}, & \beta > 2, \\ |x|^{2-\alpha}\log(1/|x|), & \beta = 2, \\ |x|^{\beta-\alpha}, & \beta \in (0,2). \end{cases}$$

Constants may depend on α and β.

Critical potentials: α-stable and relativistic α-stable

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Constants may depend on α and β.

Critical potentials: α-stable and relativistic α-stable

Following [BDK] if

$$V(x) := \frac{\bar{h}(x)}{h(x)} \qquad \left| \begin{array}{l} \bar{h}(x) = \int_0^\infty f'(t) \int_X p(t,x,y)\mu(dy)dt \\ h(x) = \int_0^\infty f(t) \int_X p(t,x,y)\mu(dy)dt \end{array} \right.$$

then

$$\int_X \tilde{p}_V(t,x,y)h(y)\, m(dy) \le h(x)\,.$$

We have

$$V_\beta^0(x) = \kappa_\beta |x|^{-\alpha} = \frac{(-\Delta)^{\alpha/2} h_\beta^0(x)}{h_\beta^0(x)}\,, \qquad h_\beta^0(x) = |x|^{-\beta}\,.$$

$$V_\beta^1(x) = \frac{(-\Delta + 1)^{\alpha/2} h_\beta^1(x)}{h_\beta^1(x)}\,, \qquad h_\beta^1(x) = \frac{2^{\frac{\beta}{2}+1}\Gamma\left(\frac{d-\beta}{\alpha}\right)}{(4\pi)^{d/2}\Gamma\left(\frac{d-\beta}{2}\right)} |x|^{-\frac{\beta}{2}} K_{\frac{\beta}{2}}(|x|)\,.$$

$f(t) = t^{\frac{d-\alpha-\beta}{\alpha}}, \quad \mu(dy) = \delta_0(dy), \quad m(dy) = dy.$

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$$V^0_\beta(x) = \kappa_\beta |x|^{-\alpha} = \frac{(-\Delta)^{\alpha/2} h^0_\beta(x)}{h^0_\beta(x)}\,, \qquad h^0_\beta(x) = |x|^{-\beta}\,.$$

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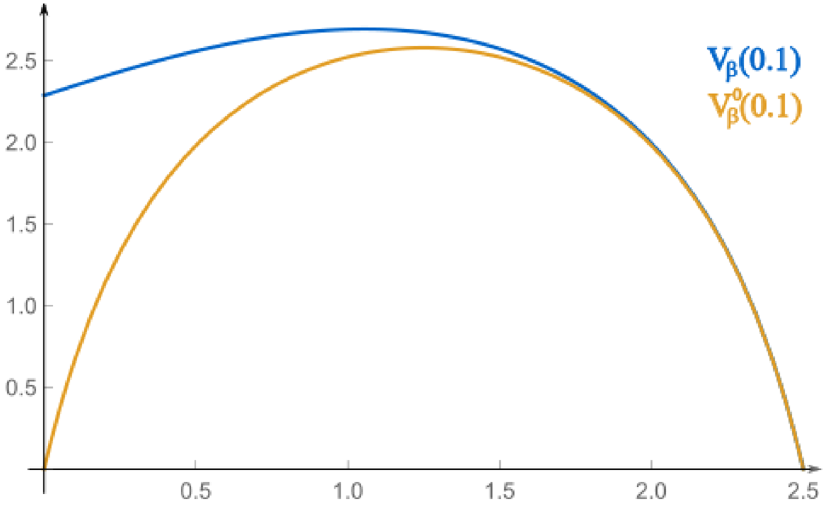
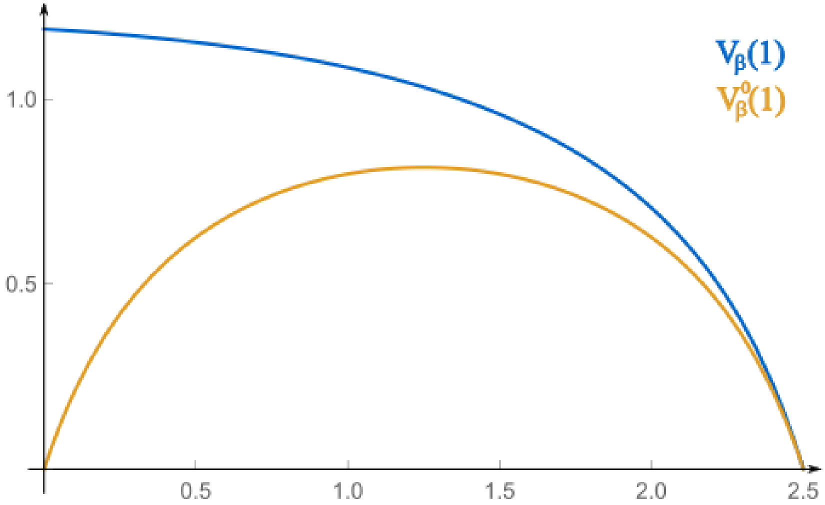
Critical potentials: α-stable and relativistic α-stable



Critical potentials: α-stable and relativistic α-stable



The four densities

Fix $\beta \in (0, \frac{d-\alpha}{2}]$ and remove it from the notation.

Ker. \ Pot.	V^0	V^1
$p^0(t,x,y)$	$\tilde{p}^0_{V^0}$	$\tilde{p}^0_{V^1}$
$p^1(t,x,y)$	$\tilde{p}^1_{V^0}$	$\tilde{p}^1_{V^1}$

Recall that $p^1 \leq p^0$ and $V^0 \leq V^1$. The density increases if $\longrightarrow$ or $\Big\uparrow$.

The function $\tilde{p}^0_{V^0}$ was studied [BGJP].

Local in time estimates

Let $\beta \in (0, \frac{d-\alpha}{2}]$ and $H^0(t,x) = 1 + t^{\beta/\alpha}|x|^{-\beta}$.

Theorem 1. Let $T > 0$. We have on $(0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ that

$$\tilde{p}^0_{V^1}(t,x,y) \approx \tilde{p}^0_{V^0}(t,x,y) \approx p^0(t,x,y) H^0(t,x) H^0(t,y)\,,$$

and

$$\tilde{p}^1_{V^1}(t,x,y) \approx \tilde{p}^1_{V^0}(t,x,y) \approx p^1(t,x,y) H^0(t,x) H^0(t,y)\,.$$

We write $f \approx g$ if there is $c \geq 1$ such that $c^{-1} \leq f(t,x,y)/g(t,x,y) \leq c$.

The criticality - blowup vs no-blowup

Recall that κ_β attains maximum at $\beta^* = \frac{d-\alpha}{2}$. For $\varepsilon > 0$ consider

$$q(x) := (1+\varepsilon)\kappa_{\beta^*}|x|^{-\alpha}.$$

Then by [BGJP]

$$\tilde{p}^0_q(t,x,y) \equiv \infty.$$

Theorem 2. Let

$$q(x) := (1+\varepsilon)V^1_{\beta^*}(x).$$

Then

$$\tilde{p}^1_q(t,x,y) \equiv \infty.$$

Theorem 3. Let $\beta < \beta^*$. There is $\varepsilon > 0$ such that if

$$q(x) := (1+\varepsilon)V^1_{\beta}(x),$$

then for all $t > 0$, $x, y \neq 0$,

$$\tilde{p}^1_q(t,x,y) < \infty.$$

Hardy's equality

Let

$$P_t u(x) = \int_{\mathbb{R}^d} p^1(t,x,y)u(y)\,dz \qquad \text{and} \qquad \mathcal{E}(u,u) = \lim_{t\to 0^+} \frac{1}{t}\langle u - P_t u, u\rangle \;.$$

Theorem 4. Let $\delta \in (0, d-\alpha)$. Then for every $u \in L^2(\mathbb{R}^d)$,

$$\mathcal{E}(u,u) = \int_{\mathbb{R}^d} u^2(x) V_\delta^1(x)\,dx$$
$$+ \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left[\frac{u(x)}{h_\delta^1(x)} - \frac{u(y)}{h_\delta^1(y)}\right]^2 h_\delta^1(y)h_\delta^1(x)\nu(|x-y|)\,dydx\,,$$

where

$$h_\delta^1(x) = c\,|x|^{-\frac{\delta}{2}} K_{\frac{\delta}{2}}(|x|)\,,$$

and

$$\nu(r) = \frac{2^{\frac{\alpha-d}{2}+1}}{|\Gamma(-\frac{\alpha}{2})|\pi^{d/2}}\, r^{-\frac{d+\alpha}{2}} K_{\frac{d+\alpha}{2}}(r)\,.$$

Thank you for your attention.