Heat kernels of non-local Schrödinger operators with Kato potentials

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(joint work with Tomasz Grzywny and Kamil Kaleta)

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Let $d \in \{1, 2, \dots\}$, and $\nu(dy) = \nu(y)\,dy$ be an absolutely continuous, symmetric Lévy measure on $\mathbb{R}^d \setminus \{0\}$, i.e.,

$$\int_{\mathbb{R}^d\setminus\{0\}} \left(1 \wedge |y|^2\right) \nu(y)\,dy < \infty, \quad \nu(-y) = \nu(y).$$

We consider non-local operators

$$\mathcal{L}\varphi(x) = \int_{\mathbb{R}^d} \left(\varphi(x+y) - \varphi(x) - \nabla\varphi(x) \cdot y \mathbf{1}_{B(0,1)}(y)\right) \nu(dy),$$

$\varphi \in C_c^2(\mathbb{R}^d)$.

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Assumptions on ν

We assume that there exists a continuous, positive and nonincreasing function f, such that $\nu(x) \asymp f(|x|)$ and

(A) There exists a constant $L_1 > 0$ such that

$$f_1 * f_1(x) \leqslant L_1\, f(x), \quad |x| \geqslant 2,$$

where $f_1(x) = \mathbf{1}_{(1,\infty)}(|x|) f(x)$.

(B) The function $s \to s^d f(s)$ is decreasing on $(0, 2]$ and there exist constants $M_1, M_2 > 0$ and $0 < \alpha_1 \leqslant \alpha_2 < 2$, such that

$$M_1 \left(\frac{R}{r}\right)^{d+\alpha_1} \leqslant \frac{f(r)}{f(R)} \leqslant M_2 \left(\frac{R}{r}\right)^{d+\alpha_2}, \quad 2 \geqslant R \geqslant r > 0.$$

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Examples

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$$f(r) = \mathbf{1}_{(0,1]}(r) \cdot r^{-\alpha-d} + e^{m}\mathbf{1}_{(1,\infty)}(r) \cdot e^{-mr^{\beta}} r^{-d-\eta},$$

where $m \geqslant 0$, $\beta \in (0,1]$, $\alpha \in (0,2)$, and

$$m = 0 \text{ and } \eta > 0,$$

or

$$m > 0 \text{ and } \beta \in (0,1) \text{ and } d+\eta \geqslant 0,$$

or

$$m > 0 \text{ and } \beta = 1 \text{ and } \frac{d-1}{2} + \eta > 0.$$

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A Borel function $q:\ \mathbb{R}^d \to \mathbb{R}$ belongs to the Kato class $\mathcal{J}$ corresponding to $\mathcal{L}$ if

$$\lim_{\delta\to 0^+} \sup_{x\in\mathbb{R}^d} \int_{|y-x|<\delta} \frac{1}{|y-x|^{2d} f(|y-x|)} |q(y)|\, dy = 0.$$

Theorem 1

Let $q \in \mathfrak{J}$ and let (A) and (B) hold. Then there exists $\widetilde{p} : (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to (0,\infty)$ such that

$$\int_0^\infty \int_{\mathbb{R}^d} \widetilde{p}(t,x,y) \left(\partial_t \phi(s+t,y) + \mathcal{L}\phi(s+t,y) + q(y)\phi(s+t,y)\right) dydt \\ = -\phi(s,x), \quad x \in \mathbb{R}^d, s > 0, \phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d).$$

$\widetilde{p}$ - jointly continuous and symmetric function of (x,y), satisfies:

$$\int_{\mathbb{R}^d} \widetilde{p}(s,x,z)\widetilde{p}(t,z,y)\, dz = \widetilde{p}(t+s,x,y), \quad x,y, \in \mathbb{R}^d, t,s > 0,$$

and $\forall_{\eta \in (0,1)} \exists_{h_\eta > 0} \forall_{m \in \mathbb{N}} \forall_{0 < t < m h_\eta}$

$$(1-\eta)^m \leqslant \frac{\widetilde{p}(t,x,y)}{p(t,y-x)} \leqslant \frac{1}{1-\eta} \exp \frac{\eta t}{h_\eta (1-\eta)}, \quad x,y \in \mathbb{R}^d, \qquad (1)$$

where p is a heat kernel of the operator $\mathcal{L}$.

There exists a corresponding to $\mathcal{L}$ probabilistic convolution semigroup of measures $\{\mu_t\}_{t\geqslant 0}$ on $\mathbb{R}^d$, such that

$$\mathcal{F}\mu_t(u) = \int_{\mathbb{R}^d} e^{iu\cdot y}\, \mu_t(dy) = e^{-t\Phi(u)}, \quad t \geqslant 0, u \in \mathbb{R}^d,$$

where

$$\Phi(u) = \int (1 - \cos(u\cdot y))\, \nu(y)\, dy, \quad u \in \mathbb{R}^d.$$

Semigroup $\{P_t,\, t \geqslant 0\}$ corresponding to $\{\mu_t,\, t \geqslant 0\}$ is given by $P_t\varphi(x) = \int \varphi(x+y)\, \mu_t(dy)$, where $\varphi \in C_\infty(\mathbb{R}^d)$. The generator:

$$\tilde{\mathcal{L}}\varphi(x) = \lim_{t\to 0^+} \frac{P_t\varphi(x) - \varphi(x)}{t},$$

$\varphi \in D = \{\varphi \in C_\infty(\mathbb{R}^d) :$
$\lim_{t\to 0^+} \frac{P_t\varphi(x)-\varphi(x)}{t}$ exists uniformly in $x \in \mathbb{R}^d.\}$.
We have $C_c^2(\mathbb{R}^d) \subset D$ and $\tilde{\mathcal{L}}\varphi = \mathcal{L}\varphi$ for all $\varphi \in C_c^2(\mathbb{R}^d)$.

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We have $\mu_t(dx) = p(t,x)dx$ and

$$\partial_t p(t,x) = \mathcal{L} p(t,x).$$

or

$$\int_0^\infty \int p(t,y-x)\left(\partial_t + \mathcal{L}\right)\phi(s+t,y)\,dydt = -\phi(s,x)$$

for every $x \in \mathbb{R}^d$, $s > 0$ and $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$.
Recall that we consider here a **Schrödinger perturbation** of $\mathcal{L}$: operator $\mathcal{L} + q$, where q is a potential from Kato class and we have a solution $\tilde{p}$ of

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K. Kaleta, P.S., *Small-time sharp bounds for kernels of convolution semigroups,* J. Anal. Math. 132 (2017).

Theorem 2

If (A) and (B) hold then there exists $t_0 > 0$ *such that*

$$p(t,x) \asymp \min\{h(t)^{-d}, tf(|x|)\}, \quad 0 < t \leqslant t_0, x \in \mathbb{R}^d,$$

where $h(t) = \frac{1}{\Psi^{-1}(1/t)}$ *and* $\Psi(s) = \sup_{|u| \leqslant s} \Phi(u)$.

K. Bogdan, W. Hansen, T. Jakubowski, *Time-dependent Schrödinger perturbations of transition densities*, Studia Mathematica 189(3) (2008).

Theorem If $p(t,x) > 0$ for all $t > 0, x \in \mathbb{R}^d$ and for every $\eta > 0$ there exists $\beta, t_0 > 0$ such that

$$\int_0^t \int_{\mathbb{R}^d} p(s,z-x)|q(z)|p(t-s,y-z)\,dzds \leqslant (\eta+\beta t)p(t,y-x),$$

for all $x, y \in \mathbb{R}^d$, $t \in (0,t_0]$, then there exists unique transition density $\tilde{p}$ locally in time comparable with p and such that

$$\tilde{p}(t,x,y) = p(t,y-x) + \int_0^t \int_{\mathbb{R}^d} p(s,z-x)q(z)\tilde{p}(t-s,z,y)\,dzdu,$$

for all $x, y \in \mathbb{R}^d$, $t > 0$.

Denoting $\tilde{P}f(t,x) = \int_0^\infty \int_{\mathbb{R}^d} \tilde{p}(s,x,z) f(t+s,z)\,dzds$,
$Pf(t,x) = \int_0^\infty \int_{\mathbb{R}^d} p(s,z-x) f(t+s,z)\,dzds$,
$Qf(t,x) = q(x)f(t,x)$, we can write **the Duhamel's formula**

$$\tilde{p}(t,x,y) = p(t,y-x) + \int_0^t \int_{\mathbb{R}^d} p(s,z-x)q(z)\tilde{p}(t-s,z,y)\,dzds,$$

in the form

$$\tilde{P} = P + PQ\tilde{P}.$$

We (should) get

$$\tilde{P} = P + PQ\tilde{P} = \tilde{P} = P + PQ(P + PQ\tilde{P})$$

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Having

$$\tilde{P} = \sum_{n=0}^{\infty} (PQ)^n P,$$

and

$$P(\partial_t + \mathcal{L}) = -Id,$$

we get

$$\tilde{P}(\partial_t + \mathcal{L} + Q) = \sum_{n=0}^{\infty} (PQ)^n P(\partial_t + \mathcal{L} + Q)$$

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The plan:

1. Verify the assumption of Theorem: $\forall_{\eta>0}, \exists_{\beta,t_0>0}, \forall_{0<t<t_0}$

$$\int_0^t \int_{\mathbb{R}^d} p(s, x-z)|q(z)|p(t-s, z-y)\, dzds \leqslant (\eta+\beta t)p(t, y-x),$$

for 'reasonable' q.

2. Make the above computation strict, proving that $\tilde{P}(\partial_t + \mathcal{L} + Q) = -Id$.

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For Lévy processes and corresponding semigroups with

$$\nu(u) \asymp |u|^{-\alpha-d}, \quad u \in \mathbb{R}^d \setminus \{0\},$$

where $\alpha \in (0,2)$, we have

$$p(t,x) \asymp \min\{t^{-d/\alpha}, t|x|^{-d-\alpha}\},$$

and 3G inequality holds:

$$p(t,x) \wedge p(s,y) \leqslant Cp(t+s, x+y), \quad t,s>0,\, x,y \in \mathbb{R}^d.$$

This yields

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and the latter integral is small if q belongs to the Kato class, because...

$$q \in \mathcal{J} \iff \lim_{\delta \to 0^+} \sup_{x \in \mathbb{R}^d} \int_{|y-x|<\delta} \frac{1}{|y-x|^{2d} f(|y-x|)} |q(y)|\, dy = 0.$$

$$\iff \lim_{\delta \to 0^+} \sup_{x \in \mathbb{R}^d} \int_{|y-x|<\delta} \frac{1}{|y-x|^{d} \Psi(|y-x|)} |q(y)|\, dy = 0.$$

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The inequality 3G

$$p(t,x) \wedge p(s,y) \leqslant Cp(t+s, x+y), \quad t,s>0,\ x,y \in \mathbb{R}^d.$$

does not hold if $p(t,x) \asymp e^{-|x|}$ (or similarly) in ∞ :

$$e^{-|x|} \wedge e^{-|x|} = e^{-|x|} \geqslant e^{-2|x|}.$$

Instead of 3G we prove directly that:

$$\int_0^t \int_{\mathbb{R}^d} p(s,z-x)|q(z)|p(t-s,y-z)\,dzds \leqslant (\eta+\beta t)p(t,y-x).$$

Let $a \geqslant 1$.

$$q \in \mathcal{J}_a \iff \lim_{\delta \to 0^+} \sup_{x \in \mathbb{R}^d} \int_{|y-x|<\delta} \frac{1}{|y-x|^{d(1+1/a)} f(|y-x|)^{1/a}} |q(y)|\, dy = 0.$$

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$$\iff \lim_{t \to 0} \sup_{x \in \mathbb{R}^d} |q(z)| \left(\int_0^t \int_{\mathbb{R}^d} [p(s, z-x)]^a \, ds \right)^{1/a} dz = 0,$$

We have $\mathcal{J}_a \subset \mathcal{J}_b$ for $a > b \geqslant 1$.

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$$\iff \lim_{\delta \to 0^+} \sup_{x \in \mathbb{R}^d} \int_{|y-x|<\delta} \frac{1}{|y-x|^{d} \Psi(|y-x|)^{1/a}} |q(y)| \, dy = 0.$$

$$\iff \lim_{t \to 0} \sup_{x \in \mathbb{R}^d} |q(z)| \left(\int_0^t \int_{\mathbb{R}^d} [p(s, z-x)]^a \, ds \right)^{1/a} dz = 0,$$

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(C) There exists a constant $C_6 > 0$ such that

$$\frac{f(r+\kappa)}{f(r)} \leqslant C_6 \frac{f(s+\kappa)}{f(s)}, \quad s > r > 0,\ \kappa > 0.$$

Let $G_n(t,x) = \min\{h(t)^{-d}, tf(\frac{|x|}{4})\} + h(t)^{-d}\left(1+\frac{|x|}{h(t)}\right)^{-n}$.

Theorem 3

If (A), (B), (C) holds and $q \in \mathcal{J}_a$ for some $a \in (1,2]$ then for every $n > 0$ we have

$$|\tilde{p}(t,z,y) - \tilde{p}(t,x,y)| \leqslant c\left(\frac{|z-x|}{h(t)} \wedge 1\right)^{\frac{\alpha_1}{b}} \left(G_n(t,y-z) + G_n(t,y-x)\right),$$

for $t \in (0,t_0), x, y \in \mathbb{R}^d$, where $\frac{1}{b} = 1 - \frac{1}{a}$.
If $\alpha_1 > 1$ and $a > \frac{\alpha_1}{\alpha_1 - 1}$ then for every $t > 0$ $\tilde{p}(t,x,y)$ has all partial derivatives w.r.t. to x, and

$$\left|\frac{\partial}{\partial x_i}\tilde{p}(t,x,y)\right| \leqslant ch(t)^{-1}G_n(t,y-x), \quad t \in (0,t_0), x, y \in \mathbb{R}^d.$$

Let

$$\widetilde{P}_t\varphi(x) = \int_{\mathbb{R}^d} \widetilde{p}(t,x,y)\varphi(y)dy, \quad t > 0.$$

Let $\beta \in (0,1)$. Denote by $C^{0,\beta}(\mathbb{R}^d)$ the space of β-Hölder continuous functions on $\mathbb{R}^d$ with its seminorm

$$\|\varphi\|_{C^{0,\beta}} := \sup_{\substack{x,y\in\mathbb{R}^d \\ x\neq y}} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^{\beta}}.$$

Theorem 4

Let (A), (B) and (C) hold.

(1) *Let* $\beta \in (0, \alpha_1/2]$. *If* $q \in \mathcal{J}_a$ *for some* $a \in [\alpha_1/(\alpha_1 - \beta), 2]$, *then for every* $t > 0$ *and* $p \in [1, \infty]$ *the operator* $\widetilde{P}_t$ *maps* $L^p(\mathbb{R}^d)$ *continuously into* $C^{0,\beta}(\mathbb{R}^d)$, *and for any* $t_0 > 0$ *there exists a constant* $C = C(p, t_0)$ *such that*

$$\left\| \widetilde{P}_t \right\|_{L^p \to C^{0,\beta}} \leqslant C h(t)^{-d/p-\beta}, \quad t \in (0, t_0).$$

(2) *If* $\alpha_1 > 1$, $q \in \mathcal{J}_a$ *for some* $a > \alpha_1/(\alpha_1 - 1)$ *and* $\varphi \in L^p(\mathbb{R}^d)$ *for some* $p \in [1, \infty]$, *then for every* $t > 0$ *the function* $\widetilde{P}_t \varphi$ *has all partial derivatives everywhere in* $\mathbb{R}^d$ *and for every* $t_0 > 0$ *there exists a constant* $C = C(p, t_0)$ *such that*

$$\left\| \nabla \widetilde{P}_t \right\|_{L^p \to L^\infty} \leqslant C h(t)^{-d/p-1}, \quad t \in (0, t_0).$$

Thank you very much for your attention!