Mehler functions and CR-extension problems

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- Heat kernels: from Ornstein-Uhlenbeck to Heisenberg subLaplacian
- The Heisenberg group
- Conformal fractional powers of the subLaplacian and modified Mehler kernels

based on the following joint works with N. Garofalo:

- *Feeling the heat in a group of Heisenberg type*, Adv. Math. (2021);
- *A heat equation approach to intertwining*, J. Anal. Math. (to appear);
- *Heat kernels for a class of hybrid evolution equations*, Potential Anal. (to appear).

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Let's start from Kolmogorov

Consider the class of differential equations in $\mathbb{R}^N \times (0, \infty)$,

$$\mathscr{A} u - \partial_t u := \mathrm{tr}(Q\nabla^2 u) + \langle Bz, \nabla u \rangle - \partial_t u = 0.$$

Here, the $N \times N$ matrices Q and B have real, constant coefficients, $Q = Q^\star \geq 0$, and

$$K(t) = \frac{1}{t}\int_0^t e^{sB} Q e^{sB^\star}\, ds \; > \; 0 \qquad \text{for all } t > 0.$$

As $t \to tK(t)$ is strictly increasing in the sense of quadratic forms, the symmetric and nonnegative definite matrix

$$K_\infty^{-1} = \lim_{t\to\infty} (tK(t))^{-1}$$

is well-defined.

If we let

$$p(z,\zeta,t)=\frac{(4\pi)^{-N/2}}{(\det(tK(t)))^{1/2}}\exp\left(-\frac{\langle K(t)^{-1}(\zeta-e^{tB}z),\zeta-e^{tB}z\rangle}{4t}\right)$$

and we denote

$$f(z,t)=P_tf_0(z)=\int_{\mathbb{R}^N}p(z,\zeta,t)f_0(\zeta)dY.$$

Then f solves the Cauchy-problem

$$\begin{cases}\mathscr{A}f-\partial_t f=0 & \text{in } \mathbb{R}^N\times(0,\infty),\\ f(z,0)=f_0(z), & \end{cases}$$

under the assumption that $f_0\in C(\mathbb{R}^N)$ satisfies

$$e^{-\frac{1}{4}\langle K_\infty^{-1}z,z\rangle}f_0(z)\in L_z^\infty\left(\mathbb{R}^N\right).$$

A two-step reduction: Example # 1

Let us make the following choice for $\mathscr{A}$

$$Q = I_N \qquad \text{and} \qquad B = -2D,$$

where $D = D^* \geq 0$. Then, if we indicate

$$j(x) = \frac{x}{\sinh(x)},$$

we have

$$(tK(t))^{-1} = \frac{1}{t} e^{tD} j(2tD) e^{tD}$$

and in particular

$$K_\infty^{-1} = 4D.$$

Let's make the following change

$$v(z,t) = e^{-\left(\frac{1}{2}\langle Dz,z\rangle + t\,\mathrm{tr}\,D\right)} f(z,t)$$

with $f = P_t f_0$, then v solves the Cauchy problem

$$\begin{cases} \Delta_z v - |Dz|^2 v - \partial_t v = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty), \\ v(z,0) = v_0(z) \quad \text{where } e^{-\frac{1}{2}\langle Dz,z\rangle} v_0(z) = e^{-\frac{1}{4}\langle K_\infty^{-1} z,z\rangle} f_0(z) \in L_z^\infty\left(\mathbb{R}^N\right). \end{cases}$$

We can write

$$v(z,t) = \int_{\mathbb{R}^N} \mathscr{M}(z,\zeta,t) v_0(\zeta) d\zeta$$

where

$$\mathscr{M}(z,\zeta,t) = (4\pi t)^{-\frac{N}{2}} \sqrt{\det j(2tD)} \exp\left\{ -\frac{1}{4t}\Big(\langle j(2tD)\cosh 2tD\ z,z\rangle + \langle j(2tD)\cosh 2tD\ \zeta,\zeta\rangle - 2\langle j(2tD)\ z,\zeta\rangle \Big)\right\}.$$

One more choice: for $\lambda \in \mathbb{R}$, we let

$$D = \pi|\lambda|\mathbb{I}_N$$

and

$$v_\lambda(z,t) = \hat{u}(z,\lambda,t) = \int_{\mathbb{R}} e^{-2\pi i\lambda\sigma} u(z,\sigma,t)d\sigma.$$

Then u solves

$$\begin{cases} \Delta_z u + \frac{|z|^2}{4}\partial_{\sigma\sigma} u - \partial_t u = 0 & \text{in } \mathbb{R}^{N+1}\times(0,\infty), \\ u((z,\sigma),0) = u_0(z,\sigma) \in \mathscr{S}(\mathbb{R}^{N+1}), \end{cases}$$

and one has

$$u((z,\sigma),t) = \int_{\mathbb{R}^N}\int_{\mathbb{R}} \mathscr{B}((z,\sigma),(\zeta,\tau),t)u_0(\zeta,\tau)d\zeta d\tau$$

where

$$\mathscr{B}((z,\sigma),(\zeta,\tau),t) = \frac{2}{(4\pi t)^{\frac{N}{2}+1}} \int_{\mathbb{R}} e^{-\frac{i}{t}\lambda(\tau-\sigma)} \left(\frac{|\lambda|}{\sinh|\lambda|}\right)^{\frac{N}{2}} \times e^{-\frac{|\lambda|}{4t\tanh|\lambda|}((|z|^2+|\zeta|^2)-2\langle z,\zeta\rangle \operatorname{sech}|\lambda|)} d\lambda.$$

The operator

$$\Delta_z u + \frac{|z|^2}{4}\partial_{\sigma\sigma} u$$

is known as Baouendi-Grushin operator.

Same two-step reduction in the degenerate case: Example # 2

Let $N = 2n$, $\lambda \in \mathbb{R}$, $z = (v, x) \in \mathbb{R}^n \times \mathbb{R}^n$, and make the following choice for $\mathscr{A}$

$$Q = \begin{pmatrix} I_n & O_{n\times n} \\ O_{n\times n} & O_{n\times n} \end{pmatrix}, \qquad B_\lambda = \begin{pmatrix} -2\pi|\lambda| I_n & O_{n\times n} \\ I_n & O_{n\times n} \end{pmatrix},$$

one has

$$tK_\lambda(t) =$$

$$e^{-2\pi t|\lambda|} \begin{pmatrix} \frac{\sinh(2\pi t|\lambda|)}{2\pi|\lambda|} I_n & \frac{\cosh(2\pi t|\lambda|)-1}{4\pi^2|\lambda|^2} I_n \\ \frac{\cosh(2\pi t|\lambda|)-1}{4\pi^2|\lambda|^2} I_n & \frac{e^{2\pi t|\lambda|}(2\pi t|\lambda| - \sinh(2\pi t|\lambda|)) + (\cosh(2\pi t|\lambda|)-1)(e^{2\pi t|\lambda|}-1)}{8\pi^3|\lambda|^3} I_n \end{pmatrix},$$

and

$$K_{\lambda,\infty}^{-1} = \begin{pmatrix} 4\pi|\lambda| I_n & O_{n\times n} \\ O_{n\times n} & O_{n\times n} \end{pmatrix}.$$

In the same spirit, the approach previously described produces the following explicit heat kernel (with pole at a generic point $(v_0, \sigma_0, x_0) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$)

$$h((v,\sigma,x),(v_0,\sigma_0,x_0),t) =$$

$$(4\pi)^{-n} \int_{\mathbb{R}} e^{2\pi i\lambda(\sigma-\sigma_0)} e^{-\frac{\pi|\lambda|}{2}(|v|^2-|v_0|^2+2nt)} (\det tK_\lambda(t))^{-1/2} \times$$

$$\exp\left\{ -\frac{1}{4}\langle (tK_\lambda(t))^{-1} \begin{pmatrix} v_0 - e^{-2\pi t|\lambda|} v \\ x_0 - x - \frac{1-e^{-2\pi t|\lambda|}}{2\pi|\lambda|} v \end{pmatrix}, \begin{pmatrix} v_0 - e^{-2\pi t|\lambda|} v \\ x_0 - x - \frac{1-e^{-2\pi t|\lambda|}}{2\pi|\lambda|} v \end{pmatrix} \rangle \right\} d\lambda$$

for the (doubly)-degenerate operator in $\mathbb{R}^{2n+1} \times \mathbb{R}_t$

$$\Delta_v + \frac{|v|^2}{4}\partial_{\sigma\sigma} + \langle v, \nabla_x \rangle - \partial_t.$$

A 'symmetric' version of Example # 1

Let us go back to

$$\Delta_z v - |Dz|^2 v - \partial_t v = 0.$$

Consider $D = \sqrt{S^* S}$ where S is a skew-symmetric $N \times N$ matrix. Under the change

$$v(z,t) = \tilde{v}(e^{-2itS} z, t),$$

one is lead to consider

$$\Delta_z \tilde{v} - |Sz|^2 \ \tilde{v} + 2i\langle Sz, \nabla_z \tilde{v}\rangle - \partial_t \tilde{v} = 0.$$

Knowing the Mehler kernel $\mathscr{M}$, we can write the heat kernel for such harmonic oscillator with complex drift.

As a matter of fact the heat kernel is given by

$$\begin{aligned}\mathscr{Q}(z,\zeta,t) &= \mathscr{M}(e^{2itS}z,\zeta,t)\\ &= \frac{e^{i\langle Sz,\zeta\rangle}}{(4\pi t)^{\frac{N}{2}}}\left(\det j\left(2t\sqrt{S^\star S}\right)\right)^{\frac{1}{2}} e^{-\frac{1}{4t}\langle j\left(2t\sqrt{S^\star S}\right)\cosh\left(2t\sqrt{S^\star S}\right)(z-\zeta),z-\zeta\rangle}.\end{aligned}$$

We notice the presence of a hidden 'translation' invariance (up to the a phase) which we now want to make more transparent.

To this aim we fix $N = 2n$ and we consider a skew-symmetric matrix such that $J^* J = I_{2n}$. For $\lambda \in \mathbb{R}$ we let

$$S = \pi \lambda J$$

and

$$\tilde{v}_\lambda(z,t) = \hat{u}(z,\lambda,t) = \int_{\mathbb{R}} e^{-2\pi i \lambda \sigma} u(z,\sigma,t) d\sigma.$$

Then u solves in $\mathbb{R}^{2n+1} \times \mathbb{R}_t$

$$\begin{aligned} 0 &= \Delta_z u + \frac{|z|^2}{4} \partial_{\sigma\sigma} u + \langle Jz, \nabla_z \partial_\sigma u \rangle - \partial_t u \\ &= \sum_{j=1}^{2n} \left(\partial_{z_j} + \frac{1}{2}(Jz)_j \partial_\sigma \right)^2 u - \partial_t u = \Delta_{\mathbb{G}} u - \partial_t u. \end{aligned}$$

The operator $\Delta_{\mathbb{G}}$ is the Heisenberg subLaplacian (or subLaplacian in the Heisenberg group).

In this way we can find the known formula for the heat kernel in the Heisenberg group

$$p((z,\sigma),(\zeta,\tau),t) = \frac{2}{(4\pi t)^{\frac{2n+2}{2}}} \int_{\mathbb{R}} e^{-\frac{i}{t}\lambda(\sigma-\tau+1/2\langle Jz,\zeta\rangle)} \left(\frac{|\lambda|}{\sinh|\lambda|}\right)^n e^{-\frac{|z-\zeta|^2}{4t}\frac{|\lambda|}{\tanh|\lambda|}}\, d\lambda$$

which was first found indipendently by Gaveau (1977) and Hulanicki (1976).

In $\mathbb{G}$ we have, respectively, the following non-Abelian multiplication law and the family of non-isotropic dilations

$$(z,\sigma)\circ(\zeta,\tau) = \left(z+\zeta, \sigma+\tau+\frac{1}{2}\langle Jz,\zeta\rangle\right),$$

$$\delta_r(z,\sigma) = (rz, r^2\sigma), \qquad \text{for } r > 0.$$

The number $2n+2$ is known as the homogeneous dimension of $\mathbb{G}$.

The fundamental solution of $\Delta_{\mathbb{G}}$ was found by Folland (1973). Fixing the pole at the group identity $e = (0,0)$, this is given by

$$\frac{2^n \Gamma^2(\frac{n}{2})}{\pi^{n+1}} \frac{1}{\left(|z|^4 + 16\sigma^2\right)^{\frac{2n+2-2}{4}}}.$$

The function $\left(|z|^4 + 16\sigma^2\right)^{\frac{1}{4}}$ is the so-called gauge function (and it is a perfect norm!).

The Heisenberg group $\mathbb{H}^n$, $n \geq 1$, is the flat model for pseudoconvex real hypersurfaces in $\mathbb{C}^{n+1}$. Celebrated works by Folland-Stein (1974), Beals-Fefferman-Grossman (1983) and Jerison-Lee (1987) highlighted the strong parallelism with the case of the flat $\mathbb{R}^N$ in Riemannian/Conformal geometry. In such parallelism the role of the Laplacian is taken in $\mathbb{H}^n$ by $\Delta_{\mathbb{G}}$.

Many radial fundamental solutions

In Euclidean $\mathbb{R}^N$, with $N > 2s$, the fundamental solution of the s-Laplacian $(-\Delta)^s$ with pole at $x_0 \in \mathbb{R}^N$ is given by

$$\frac{\Gamma\left(\frac{N}{2} - s\right)}{4^s \pi^{\frac{N}{2}} \Gamma(s)} \frac{1}{|x - x_0|^{N-2s}}.$$

Denoting

$$P_t^\Delta u(x) = \int_{\mathbb{R}^N} \frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{|x-x_0|^2}{4t}} u(x_0) dx_0,$$

we recall that, for $0 < s < 1$,

$$(-\Delta)^s u(x) = \frac{-s}{\Gamma(1-s)} \int_0^\infty \frac{P_t^\Delta u(x) - u(x)}{t^{1+s}} dt$$

and

$$\mathscr{I}^{(2s)}u(x) = \frac{1}{\Gamma(s)}\int_0^\infty \tau^{s-1} P_\tau^\Delta u(x) d\tau.$$

Having in mind that $(-\Delta)^s$ and $\mathscr{I}^{(2s)}$ invert each other, the fundamental solution of $(-\Delta)^s$ is nothing but the kernel defining $\mathscr{I}^{(2s)}$, i.e.

$$\frac{1}{\Gamma(s)}\int_0^\infty \tau^{s-1}\frac{1}{(4\pi\tau)^{\frac{N}{2}}} e^{-\frac{|x-x_0|^2}{4\tau}} d\tau = \frac{\Gamma\left(\frac{N}{2}-s\right)}{4^s\pi^{\frac{N}{2}}\Gamma(s)} \frac{1}{|x-x_0|^{N-2s}}.$$

The pure powers of the subLaplacian $\Delta_{\mathbb{G}}$ do not possess the right conformal properties (see, e.g., Graham (1984)). We have in fact the following

Proposition

For any $0 < s < 1$ the fundamental solution of the operator $(-\Delta_{\mathbb{G}})^s$ with pole at the group identity 0 is given by the formula

$$\frac{2^{1-2s}\Gamma(n+1-s)}{\pi^{\frac{2n+2}{2}}\Gamma(s)} \frac{1}{|z|^{2(n+1-s)}} \int_0^1 (\tanh^{-1}\sqrt{y})^{s-1}(1-y)^{\frac{n}{2}-1} y^{-\frac{s}{2}} \times$$
$$\times F\left(\frac{1}{2}(n+1-s), \frac{1}{2}(n+2-s); \frac{1}{2}; -\frac{16\sigma^2}{|z|^4} y\right) dy.$$

In particular, such function depends only on $|z|^4$ and σ^2, but it is not a function of the gauge $(|z|^4 + 16\sigma^2)^{1/4}$.

This means that $(-\Delta_{\mathbb{G}})^s$ is not the operator we want to work with.

We are in fact interested in the conformal fractional power of the sub-Laplacian on $\mathbb{G}$, which we denote by $\mathscr{L}_s$. This is a pseudo-differential operator which has been defined in the literature through the following spectral representation

$$\mathscr{L}_s = 2^s |\partial_\sigma|^s \frac{\Gamma(-\frac{1}{2}\Delta_{\mathbb{G}}|\partial_\sigma|^{-1} + \frac{1+s}{2})}{\Gamma(-\frac{1}{2}\Delta_{\mathbb{G}}|\partial_\sigma|^{-1} + \frac{1-s}{2})}.$$

see Branson-Fontana-Morpurgo (2013) and Frank-Lieb (2012).
We should also refer to Knapp-Stein (1971), Cowling (1980) and Cowling-Haagerup (1989) for the 'Fourier analysis' of the inverse of $\mathscr{L}_s$.

Our starting point has been the analysis developed in two papers by Frank-del Mar González-Monticelli-Tan (2015) and Roncal-Thangavelu (2016).

Heat kernels and extension problems

Let's go back to the case of $\mathbb{R}^N$, by recalling the Caffarelli-Silvestre extension operator in $\mathbb{R}^N \times \mathbb{R}^+ \ni (x, y)$: the problem

$$\begin{cases} \frac{\partial^2 U}{\partial y^2} + \frac{1-2s}{y}\frac{\partial U}{\partial y} + \Delta U = 0, \\ U(x,0) = f(x). \end{cases}$$

yields an alternative definition of $(-\Delta)^s$ through the Dirichlet-to-Neumann condition

$$-\frac{2^{2s-1}\Gamma(s)}{\Gamma(1-s)} \lim_{y \to 0^+} y^{1-2s} \partial_y U(x,y) = (-\Delta)^s f(x).$$

Let us now consider the parabolic extension problems

$$\frac{\partial^2 U}{\partial y^2} + \frac{1 \mp 2s}{y}\frac{\partial U}{\partial y} + \Delta U - \partial_t U.$$

The heat kernels (with pole at the point $(x_0, 0)$ in the thick space $\mathbb{R}^N \times \mathbb{R}_y^+$) of the above parabolic operators are given by

$$q^{(\pm s)}(x, x_0, t, y) = (4\pi t)^{-(1\mp s)} e^{-\frac{y^2}{4t}} p(x, x_0, t)$$
$$= \frac{1}{(4\pi t)^{\frac{N}{2}+1\mp s}} e^{-\frac{|x-x_0|^2+y^2}{4t}}.$$

We notice the following obvious fact

$$(4\pi t)^{(1-s)} q^{(s)}(x, x_0, t, 0) = (4\pi t)^{(1+s)} q^{(-s)}(x, x_0, t, 0) = p(x, x_0, t),$$

together with the fact that the 'same' heat semigroup P_t shows up in both the heat kernel representations for the s-Laplacian $(-\Delta)^s$ and its inverse $\mathscr{I}^{(2s)}$. This will be in striking contrast with the non-standard settings of the Heisenberg group.

What is the *right* pde underlying the conformal fraction powers of the subLaplacian in $\mathbb{H}^n$

$$\mathscr{L}_s = 2^s|\partial_\sigma|^s \frac{\Gamma(-\frac{1}{2}\Delta_{\mathbb{G}}|\partial_\sigma|^{-1} + \frac{1+s}{2})}{\Gamma(-\frac{1}{2}\Delta_{\mathbb{G}}|\partial_\sigma|^{-1} + \frac{1-s}{2})} \qquad ??$$

Frank-del Mar González-Monticelli-Tan (2015) found that the following geometric extension problem in $\mathbb{H}^n \times \mathbb{R}^+$

$$\begin{cases} \frac{\partial^2 U}{\partial y^2} + \frac{1-2s}{y}\frac{\partial U}{\partial y} + \frac{y^2}{4}\partial_{\sigma\sigma} + \Delta_{\mathbb{G}} U = 0, \\ U(z,\sigma,0) = f(z,\sigma), \end{cases}$$

provides an alternative definition of $\mathscr{L}_s$ through the Dirichlet-to-Neumann condition

$$-\frac{2^{2s-1}\Gamma(s)}{\Gamma(1-s)} \lim_{y\to 0^+} y^{1-2s}\partial_y U(z,\sigma,y) = \mathscr{L}_s f(z,\sigma).$$

As before, we now consider the parabolic extension problems

$$\frac{\partial^2 U}{\partial y^2} + \frac{1 \mp 2s}{y}\frac{\partial U}{\partial y} + \frac{y^2}{4}\partial_{\sigma\sigma} + \Delta_{\mathbb{G}} U - \partial_t U.$$

Can we write down the heat kernels of such hybrid equations?

YES: the steps outlined at the beginning of the talk are good enough to provide an explicit expression. As a matter of fact,

$$\frac{\partial^2 U}{\partial y^2} + \frac{1 - 2s}{y}\frac{\partial U}{\partial y} + \frac{y^2}{4}\partial_{\sigma\sigma} + \Delta_{\mathbb{G}} U - \partial_t U$$
$$= \frac{\partial^2 U}{\partial y^2} + \frac{1 - 2s}{y}\frac{\partial U}{\partial y} + \Delta_z U + \frac{|z|^2 + y^2}{4}\partial_{\sigma\sigma} +$$
$$+ \langle Jz, \nabla_z \partial_\sigma U\rangle - \partial_t U.$$

As before, we now consider the parabolic extension problems

$$\frac{\partial^2 U}{\partial y^2} + \frac{1 \mp 2s}{y}\frac{\partial U}{\partial y} + \frac{y^2}{4}\partial_{\sigma\sigma} + \Delta_{\mathbb{G}} U - \partial_t U.$$

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$$\begin{aligned}&\frac{\partial^2 U}{\partial y^2} + \frac{1 - 2s}{y}\frac{\partial U}{\partial y} + \frac{y^2}{4}\partial_{\sigma\sigma} + \Delta_{\mathbb{G}} U - \partial_t U\\ &= \frac{\partial^2 U}{\partial y^2} + \frac{1 - 2s}{y}\frac{\partial U}{\partial y} + \Delta_z U + \frac{|z|^2 + y^2}{4}\partial_{\sigma\sigma} +\\ &+ \langle Jz, \nabla_z \partial_\sigma U\rangle - \partial_t U.\end{aligned}$$

By applying the partial Fourier transform of U with respect to the variable $\sigma \in \mathbb{R}$, with dual variable $\lambda \in \mathbb{R}$, one is lead to consider

$$\partial_{yy} v + \frac{1 \mp 2s}{y} \partial_y v + \Delta_z v + 2\pi\lambda i \langle Jz, \nabla_z v \rangle - \pi^2 |\lambda|^2 (|z|^2 + y^2) v - \partial_t v$$

$(v(z, y, t) = \hat{U}(z, \lambda, y, t)$ where λ is fixed). Thus we can recognize that

Theorem

The heat kernels $q^{(\pm s)}(g, g', t, y) = q^{(\pm s)}((g')^{-1} \circ g, 0, t, y)$ (with pole at the point $(g', 0)$ in the thick space $\mathbb{H}^n \times \mathbb{R}^+_y$) can be written as

$$q_{(\pm s)}((z, \sigma), 0, t, y) =$$
$$\frac{2}{(4\pi t)^{n+2\mp s}} \int_{\mathbb{R}} e^{-\frac{i}{t}\lambda\sigma} \left(\frac{|\lambda|}{\sinh|\lambda|} \right)^{n+1\mp s} e^{-\frac{|z|^2+y^2}{4t} \frac{|\lambda|}{\tanh|\lambda|}} d\lambda.$$

A heat kernel look at $\mathscr{L}_s$ and its inverse

We can now define

$$\mathscr{K}_{(\pm s)}(g, g', t) = (4\pi t)^{1\mp s} q_{(\pm s)}(g, g', t, 0)$$

and

$$P_{(\pm s),t} u(g) = \int_{\mathbb{H}^n} \mathscr{K}_{(\pm s)}(g, g', t) u(g') dg'.$$

These two modified semigroups (which are not semigroups... and do not coincide!) allows us to define

$$\mathscr{L}_s u(g) = -\frac{s}{\Gamma(1-s)} \int_0^\infty \frac{1}{t^{1+s}} \left[P_{(-s),t} u(g) - u(g) \right] dt$$

and

$$\mathscr{I}_{(2s)} u(g) = \frac{1}{\Gamma(s)} \int_0^\infty \tau^{s-1} P_{(s),\tau} u(g) d\tau.$$

We have proved that the kernel of the operator $\mathscr{I}_{(2s)}$ is gauge-symmetric, i.e.

Theorem

For any $(z,\sigma) \in \mathbb{H}^n$ *we have*

$$\frac{1}{\Gamma(s)} \int_0^\infty \tau^{s-1} \mathscr{K}_{(s)}((z,\sigma),0,\tau) d\tau = \frac{2^{n+1-3s}\Gamma^2(\frac{1}{4}(2n+2-2s))}{\pi^{\frac{2n+2}{2}}\Gamma(s)} \frac{1}{(|z|^4+16\sigma^2)^{\frac{2n+2-2s}{4}}},$$

Semigroup inversion: $\mathscr{I}^{(2s)} \circ (-\Delta)^s = Id$

We can rewrite

$$\frac{-s}{\Gamma(1-s)} \int_0^\infty t^{-s-1} \left(P_t u(x) - u(x)\right) dt = -\frac{1}{\Gamma(1-s)} \int_0^\infty t^{-s} \Delta P_t u(x) dt.$$

We then obtain

$$\mathscr{I}^{(2s)} \left((-\Delta)^s u\right)(x) = -\frac{1}{\Gamma(s)\Gamma(1-s)} \int_0^\infty \int_0^\infty \frac{\tau^{s-1}}{t^s} \Delta P_{t+\tau} u(x) dt d\tau$$

The change of variable $(v, \rho) = \left(t + \tau, \frac{\tau}{t}\right)$ now yields

$$\mathscr{I}^{(2s)} \left((-\Delta)^s u\right)(x) = \frac{-1}{\Gamma(s)\Gamma(1-s)} \int_0^\infty \frac{\rho^{s-1}}{1+\rho} d\rho \int_0^\infty \Delta P_v u(x) dv = u(x)$$

since $\Delta P_v = \partial_v P_v$ and

$$\int_0^\infty \frac{\rho^{s-1}}{1+\rho} d\rho = \Gamma(s)\Gamma(1-s).$$

The semigroup property for the composition $P_{(s),\tau} \circ P_{(-s),t}$ is lost: as a matter of fact we have the following

Lemma

Fix $s \in (0,1)$, $(z,\sigma) \in \mathbb{H}^n$ and $t, \tau > 0$. Then, we have

$$\int_{\mathbb{H}^n} \mathscr{K}_{(-s)}((z,\sigma),(z',\sigma'),\tau)\mathscr{K}_{(s)}((z',\sigma'),0,t)\,dz'd\sigma'$$

$$= \int_{\mathbb{R}} e^{2\pi i\lambda\sigma} \left(\frac{2\pi t|\lambda|}{\sinh 2\pi t|\lambda|}\right)^{1-s} \left(\frac{2\pi\tau|\lambda|}{\sinh 2\pi\tau|\lambda|}\right)^{1+s} \times$$

$$\times \left(\frac{|\lambda|}{2\sinh 2\pi(t+\tau)|\lambda|}\right)^{n} e^{-\frac{\pi}{2}|z|^2 \frac{|\lambda|}{\tanh 2\pi(t+\tau)|\lambda|}}\,d\lambda.$$

Still.. the dependence with respect to $\frac{\tau}{t}$ makes no contribution in average:

Lemma

For any $s \in (-1,1)$ *and* $\mu > 0$ *we have*

$$A(s,\mu) \stackrel{def}{=} \int_0^\infty \rho^{s-1} \left[\frac{(1+s)\rho}{1+\rho} \left(\frac{\frac{\mu}{1+\rho}}{\tanh \frac{\mu}{1+\rho}} - 1 \right) - \frac{1-s}{1+\rho} \left(\frac{\frac{\rho\mu}{1+\rho}}{\tanh \frac{\rho\mu}{1+\rho}} - 1 \right) \right] \times$$

$$\times \left(\frac{\frac{\rho\mu}{1+\rho}}{\sinh \frac{\rho\mu}{1+\rho}} \right)^{1-s} \left(\frac{\frac{\mu}{1+\rho}}{\sinh \frac{\mu}{1+\rho}} \right)^{1+s} d\rho \ = \ 0.$$

Theorem

For every $0 < s < 1$ *and* $u \in C_0^\infty(\mathbb{H}^n)$ *one has*

$$\left(\mathscr{I}_{(2s)} \circ \mathscr{L}_s\right) u = \left(\mathscr{L}_s \circ \mathscr{I}_{(2s)}\right) u = u.$$

This is saying in particular that

$$\frac{2^{n+1-3s}\Gamma^2(\frac{1}{4}(2n+2-2s))}{\pi^{\frac{2n+2}{2}}\Gamma(s)} \frac{1}{(|z|^4 + 16\sigma^2)^{\frac{2n+2-2s}{4}}}$$

is the fundamental solution of the operator $\mathscr{L}_s$ with pole at the group identity.

We can go on and define

$$\mathfrak{e}_{(\pm s)}(g, g', y) \overset{def}{=} \int_0^\infty q_{(\pm s)}(g, g', t, y)dt.$$

Theorem

Let $0 < s < 1$, $g \in \mathbb{H}^n$, $y > 0$. We have

$$\mathfrak{e}_{(\pm s)}(g, 0, y) = \frac{2^{n \mp s - 1}\Gamma^2(\frac{1}{4}(2n + 2 \mp 2s))}{\pi^{\frac{2n+4\mp 2s}{2}}} \times$$
$$\times \left((|z|^2 + y^2)^2 + 16\sigma^2\right)^{-\frac{1}{4}(2n+2\mp 2s)}.$$

Theorem

Let $s \in (0,1)$. For every $g, g' \in \mathbb{H}^n$ and $y > 0$ one has

$$\mathscr{L}_s(\mathfrak{e}_{(s)}(\cdot, g', y))(g) = (2\pi y)^{2s} \mathfrak{e}_{(-s)}(g, g', y).$$

These two facts imply

$$\mathscr{L}_s\left(\left(\frac{16y^2}{(|z|^2+y^2)^2+16\sigma^2}\right)^{\frac{2n+2-2s}{4}}\right) = \frac{\Gamma^2\left(\frac{2n+2+2s}{4}\right)}{\Gamma^2\left(\frac{2n+2-2s}{4}\right)}\left(\frac{16y^2}{(|z|^2+y^2)^2+16\sigma^2}\right)^{\frac{2n+2+2s}{4}}.$$

Thanks for the attention