Regularity and upper density estimates for the law of an stable process and its supremum: From simulation to theory

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Stable process

A general α-stable process starting at 0 is a Lévy process with characteristic function ($\alpha \in (0,2)$)

$$-\log\left(\mathbb{E}\left[e^{i\theta X_t}\right]\right) = ct|\theta|^\alpha(1 - i\,\mathbf{sgn}(\theta)\tan(\pi\alpha(2\rho - 1)/2))$$

•Here, $\rho := \mathbb{P}(X_t \geq 0) \in [1 - 1/\alpha, 1/\alpha] \cap (0,1)$ is the positivity parameter. If $\rho = 1/2$ then the above simplifies to the symmetric stable case and its generator is the fractional Laplacian.

• Our goal: Study of the joint law of $(X_T, \bar{X}_T) \equiv (X_T, \sup_{s\in[0,T]} X_s)$ on the domain $\mathcal{O} := \{(x,y) \in \mathbb{R}^2; y \geq x \vee 0\}$ (including the behavior close to the diagonal and for $x, y \approx 0$ or $x, y \approx \infty$)

• In PDE terms: For a test function f, let $u_t(x,m) := \mathbb{E}_x[f(X_t, m + \bar{X}_T)]$ then for $\nu_\alpha(dz) = |z|^{-1-\alpha}dz$ in the case $\rho = 1/2$

$$\partial_t u_t(x,m) = \int\left(u_t(x+z, m + z1_{[m-x,\infty)}(z)) - u_t(x,m) - \nabla u_t(x,m)\cdot(z, z1_{[m-x,\infty)}(z))\right)\nu_\alpha(dz)$$

with the boundary conditions $\partial_m^{\alpha-1} u_t(x,-x) = 0$ and $u(0,x,m) = f(x,m)$.

Previous results

•Many previous results: R.A. Doney and M.S. Savov (2010).

$$\partial_y^3 \mathbb{P}(\bar{X}_T \in dy) \overset{y\to\infty}{\approx} Ty^{-\alpha-1}; \quad \partial_y^3 \mathbb{P}(\bar{X}_T \in dy) \overset{y\to 0}{\approx} T^{-\rho} y^{\alpha\rho-1}.$$

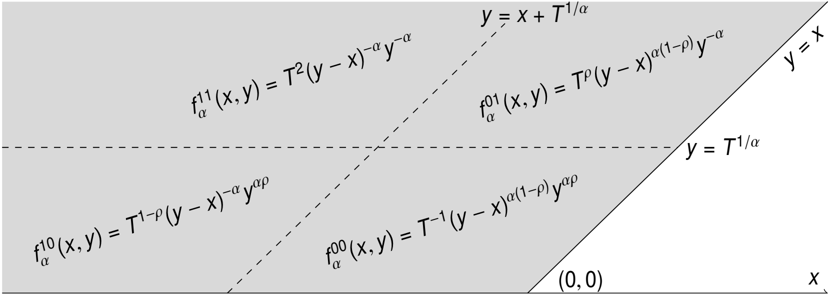
$$\partial_y^3 \mathbb{P}(X_T \in dx) \overset{x\to\infty}{\approx} Tx^{-\alpha-1}; \quad \mathbb{P}(X_T \in dx) \overset{x\to 0}{\approx} T^{-1/\alpha}.$$

Analytical arguments based on stable meanders (excursion identities, Wiener-Hopf)

• 4 domains required. Let $\tilde{F}(x, y) := \mathbb{P}(\overline{X}_T \le x, \overline{X}_T - X_T \le z)$. We will prove

$$|\partial_x^n \partial_z^m \tilde{F}(x, z)| \le C x^{-n} \min\{Tx^{-\alpha}, T^{-\rho} x^{\alpha\rho}\} \times z^{-m} \min\{Tz^{-\alpha}, T^{-(1-\rho)} z^{\alpha(1-\rho)}\}.$$

The density of $(X_T, \overline{X}_T)$ is bounded by: $y^{-1}(y-x)^{-1} f_\alpha^{ij}(x,y)$



The set $\mathcal{O} = \{(x, y) \in \mathbb{R}^2 : y > \max\{x, 0\}\}$ (shaded in the figure) is the support of the joint density of $(X_T, \overline{X}_T)$. According to our Theorem, the support can be partitioned into 4 sub-regions according to which of the functions f_α^{ij}, $i, j \in \{0, 1\}$, is the smallest in the (optimal) case $\alpha' = \alpha$. The dotted lines correspond to the change in time regime.
The above bounds are in accordance with known asymptotic results.

Main statement

Let $\mathcal{O} = \{(x,y) \in \mathbb{R}^2 : y > x \vee 0\}$, $n, m \geq 1$, $T > 0$.

Theorem

Let $F(x,y) := \mathbb{P}(X_T \leq x, \overline{X}_T \leq y)$ be the law of $(X_T, \overline{X}_T)$. Then $F \in C^\infty(\mathcal{O})$. Moreover, for any fixed $\alpha' \in [0, \alpha)$ there is $C > 0$ s.t. for all $(x,y) \in \mathcal{O}$,

$$\begin{aligned}
|\partial_x^n \partial_y^m F(x,y)| &\leq C y^{-m} (y-x)^{1-n-m} (2y-x)^{m-1} \\
&\quad \times \min\{f_{\alpha'}^{00}(x,y), f_{\alpha'}^{01}(x,y), f_{\alpha'}^{10}(x,y), f_{\alpha'}^{11}(x,y)\},
\end{aligned}$$

$$\begin{aligned}
C \min\Big\{ & T^{2\frac{\alpha'}{\alpha}} (y-x)^{-n-\alpha'} y^{-m-\alpha'}, T^{\frac{\alpha'}{\alpha}(1-\rho)} (y-x)^{-n-\alpha'} y^{-m+\alpha'\rho}, \\
& T^{\frac{\alpha'}{\alpha}\rho} (y-x)^{-n+\alpha'(1-\rho)} y^{-m-\alpha'}, T^{-\frac{\alpha'}{\alpha}} (y-x)^{-n+\alpha'(1-\rho)} y^{-m+\alpha'\rho} \Big\}.
\end{aligned}$$

4 Domains: Behavior at ∞. Behavior at 0. This result is optimal in time and almost optimal in space taking $\alpha' \approx \alpha$. The main reason is that there is some Chebyshev's type argument used.

Some consequences

•Assume that $\alpha \in (0,2)$. Then the distribution function $F(x) := \mathbb{P}(\overline{X}_T \leq x) \in C^\infty(0,\infty)$ and, for every $\alpha' \in [0,\alpha)$ and $n \geq 1$, there exists some constant $C > 0$ such that for all $x > 0$ and $T > 0$, we have for $n \geq 1$

$$|\partial_x^n F(x)| \leq Cx^{-n} \min\{T^{\frac{\alpha'}{\alpha}} x^{-\alpha'}, T^{-\frac{\alpha'}{\alpha}\rho} x^{\alpha'\rho}\}.$$

•Define $\tau_L := \inf\{t > 0 : X_t > L\}$. Then the density of τ_L is smooth and the following estimate is satisfied for $n \geq 1$:

$$|\partial_t^n \mathbb{P}(\tau_L \leq t)| \leq Ct^{-\frac{1}{\alpha}-n} \min\{L^{-\alpha'} t^{\frac{\alpha'}{\alpha}}, L^{\alpha'\rho} t^{-\frac{\alpha'\rho}{\alpha}}\}.$$

• If $\alpha(1-\rho) \geq 1$ there is no blow up of the density at the boundary $y = x$. Blow up appears with density derivatives.

• One can also obtain bounds for the derivatives of (related to overshooting) $y \geq T^{1/\alpha}$, $x \leq 0$. Then for any $\alpha' \in (0,\alpha)$

$$\mathbb{P}(X_T \leq x, \tau_y < T) \leq CT^{2\frac{\alpha}{\alpha'}} y^{-\alpha'} \times \min\{y^{-\alpha'}, (-x)^{-\alpha'}\}$$

Interpolation

Finite dimensional calculus based on a indep. increment process X uses

$$D_s = \frac{\partial}{\partial \Delta X_s}$$

Are there any other variations of this definition?
If the functional $F(X)$ can be approximated using functions F_n

$$F_n(E_1, \ldots, E_n, G_1, \ldots, G_n, U_1, \ldots, U_n) \stackrel{\mathcal{L}}{\Rightarrow} F(X)$$

One may try to do the analysis based on the i.i.d. sequence $\{E_i, G_i, U_i\}_{i \in \mathbb{N}}$ is densities are explicit.
Dangers in this method:

- The convergence rate is not good enough. R.N. Bhattacharaya and R. Ranga Rao, Normal Approximation and Asymptotic Expansions. SIAM Classics, 2010
- The used variables $\{E_i\}_{i \in \mathbb{N}}$ are not explaining the main density behavior of $F(X)$
- Because one concentrates on $F(X)$ one may loose track of all path behavior

Interpolation techniques

Can it be used?

- Fournier-Printems, Bally, V. and Caramellino: L. Stochastic Integration by Parts. Advanced Courses in Mathematics - CRM Barcelona, 2016
- Key questions:

1)How to choose the approximations?

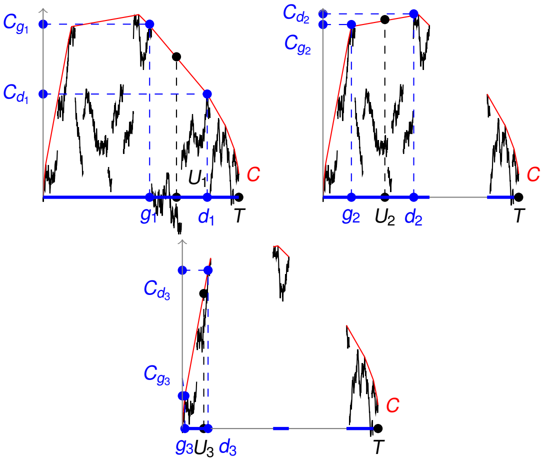
2) Which random variables to use? (Partial Malliavin Calculus)

- Main answers:

1) Convex majorants of Lévy processes (50's∼) with multi-level (2008)

2) Chambers-Mallows-Stuck decomposition method (recall that explicit stable laws are not available)('76). Base the calculations on exponential r.v.'s $\{E_i\}_{i\in\mathbb{N}}$. That is, the "length" of stable increments.

Convex majorants



Selecting the first three faces of the concave majorant: the total length of the thick blue segment(s) on the abscissa equal the stick sizes T, $T-(d_1-g_1)$ and $T-(d_1-g_1)-(d_2-g_2)$, respectively. The independent random variables U_1, U_2, U_3 are uniform on the sets $[0,T]$, $[0,T]\setminus(g_1,d_1)$, $[0,T]\setminus\bigcup_{i=1}^{2}(g_i,d_i)$, respectively. Note that the residual length of unsampled faces after n samples is L_n.

Mathematical Definition

Exponentially converging stick-breaking process: $\ell = (\ell_k)_{k\geq 1}$ on $[0, T]$, defined using $U_k \sim U[0,1]$

$$\ell_1 = T(1 - U_1)$$

$$\ell_k = TU_1 ... U_{k-1}(1 - U_k)$$

⚠ $$\mathbb{E}[\ell_k^{r/\alpha}] = T^r \left(1 + \frac{r}{\alpha}\right)^{-k}.$$

i.i.d. stable r.v.'s $(S_k)_{k\geq 1}$ with parameters (α, ρ) (i.e. $S_k \overset{d}{=} X_1$).

$$\overline{X}_T = X_+ = \sum_{k=1}^{\infty} \ell_k^{1/\alpha} [S_k]^+$$

$$X_T = X_+ - X_- = X_+ - \sum_{k=1}^{\infty} \ell_k^{1/\alpha} [S_k]^-.$$

Remark: One looses some information about the path!

Chambers-Mallows-Stuck

Next step: Probabilistic representation:(Chambers-Mallows-Stuck, $\alpha \in (0,1) \cup \{1\} \cup (1,2)$). E_k: length. G_k: oscillations

$$S_k \stackrel{\mathcal{L}}{=} E_k^{1-1/\alpha} G_k \qquad \text{and} \qquad G_k = g(V_k), \qquad k \in \mathbb{N},$$

for i.i.d. $Exp(1) \sim (E_k)_{k \geq 1} \perp (V_k)_{k \geq 1} \sim \mathrm{U}(-\frac{\pi}{2}, \frac{\pi}{2})$

$$g(x) = \frac{\sin\left(\alpha(x + \pi(\rho - \frac{1}{2}))\right)}{\cos^{1/\alpha}(x)\cos^{1-1/\alpha}\left((1-\alpha)x - \alpha\pi(\rho - \frac{1}{2})\right)}, \qquad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Note that indeed $\mathbb{P}(S_k > 0) = \rho$.

$$\rho = \mathbb{P}(S_k > 0) \in [1 - 1/\alpha, 1/\alpha] \cap (0,1).$$

semi-linear structure in the representation for $(X_T, \bar{X}_T) \equiv (X_+, X_-)$

$$\overline{X}_T = X_+ = \sum_{k=1}^{\infty} \ell_k^{1/\alpha} E_k^{1-1/\alpha} [G_k]^+$$

$$X_T = X_+ - X_- = X_+ - \sum_{k=1}^{\infty} \ell_k^{1/\alpha} E_k^{1-1/\alpha} [G_k]^-.$$

coordinate change and n-th order approx. to $X = (X_+, X_-)$:

$X_n = (X_{+,n}, X_{-,n})$

$a_n := T^{1/\alpha} \kappa^n$ with $\kappa \in (0,1)$. Let η_+ and $\eta_- \sim Exp(1)$

$$X_{\pm,n} = \sum_{k=1}^{n} \ell_k^{1/\alpha} E_k^{1-1/\alpha} [G_k]^{\pm} + a_n \eta_+^{1-1/\alpha}.$$

$n = 0$, $X_{\pm,0} \equiv 0$, $[x]^+ = \max\{x, 0\}$,

$$0 \neq X_{\pm,n} - X_{\pm,n-1} = \ell_n^{1/\alpha} E_n^{1-1/\alpha} [G_n]^{\pm} + (a_n - a_{n-1}) \eta_{\pm}^{1-1/\alpha}$$

Intuitively, if the sequence $\{a_n\}$ decays too fast, then it will not serve its purpose. In particular, given the assumption below moments estimates will follow.

Assumption[A-κ] $a_n := T^{1/\alpha} \kappa^n$, $n \in \mathbb{N}$ where $\kappa^\alpha \in [\rho \vee (1-\rho), 1)$.

Reconstructive derivative operator and IBP

$$\mathcal{D}_n^{\pm} = \eta_{\pm}\partial_{\eta_{\pm}} + \sum_{k=1}^{n} E_k \mathbf{1}_{\{[G_k]^{\pm}>0\}}\partial_{E_k},$$

⚠The factor $(\eta_{\pm}, E_k)$ cancels boundary terms and has a regenerative property for $\mathcal{D}_n^{\pm}X_{\pm,n} = (1-1/\alpha)X_{\pm,n}$. This keeps the stable moments controlled! . It also shows that G_k does not need to be differentiated.

$$\mathcal{S}_n(\Omega) = \{\Phi \in L^0(\Omega); \exists\phi \in \mathcal{S}_\infty((0,\infty)^{3n+3};\mathbb{R}), \Phi = \phi(\bar{E}_n, \bar{U}_n, \bar{V}_n, \eta_+, \eta_-)\},$$

Theorem (The approx. but exploding IBP formula)

Fix $n, m \in \mathbb{N}$ with $m \geq n$. Then for any $\Phi \in \mathcal{S}_m(\Omega)$ and f smooth,

$$\mathbb{E}[\partial_{\pm} f(X_n)\Phi] = \mathbb{E}[f(X_n)H_{n,m}^{\pm}(\Phi)],$$

$$H_{n,m}^{\pm}(\Phi) := \frac{\alpha/(\alpha-1)}{X_{\pm,n}}\left(\left(\eta_{\pm} - \frac{1}{\alpha} + \sum_{k=1}^{m}(E_k-1)\mathbf{1}_{\{[G_k]^{\pm}>0\}}\right)\Phi - \mathcal{D}_m^{\pm}[\Phi]\right).$$

m: number of variables used for the IBP

n: approximation parameter

⚠The numerator grows polynomically fast w.r.t. m.→ Explosion

⚠ Note that time rescaling is clear in the denominator. Malliavin variance

Multi-level in IBP as an interpolation method

$$H_{n,m}^{\pm,k+1}(\Phi) = H_{n,m}^{\pm}(H_{n,m}^{\pm,k}(\Phi)) \qquad \text{for } k \geq 0, \text{ where} \qquad H_{n,m}^{\pm,0}(\Phi) = \Phi.$$

Theorem (The ∞-dim. IBP formula)

Let $\Phi \in \bigcap_{n\in\mathbb{N}} \mathcal{S}_n(\Omega)$. For any $n \geq 1$, $k_+, k_- \geq 0$ and $f \in C_\epsilon^{k_+ + k_-}(\mathbb{R}_+^2)$ we have

$$\mathbb{E}[\partial_+^{k_+}\partial_-^{k_-} f(X_+, X_-)\Phi] = \mathbb{E}[f(X_n)H_{n,n}^{+,k_+}(H_{n,n}^{-,k_-}(\Phi))] + \sum_{j=n}^{\infty} \mathbb{E}[f(X_{j+1})H_{j+1,j+1}^{+,k_+}(H_{j+1,j+1}^{-,k_-}(\Phi)) - f(X_j)H_{j,j+1}^{+,k_+}(H_{j,j+1}^{-,k_-}(\Phi))].$$

⚠ The convergence in the above sum is geometric in j due to the difference $X_{j+1} - X_j$.

⚠ The weight $H_{j,j+1}^{+,k_+}$ only grows polynomically in j.

⚠ Positive and negative moment estimates are needed!

⚠ To correctly balance these terms one uses basic interpolation inequalities

The moment estimates

Let $f(x_+, x_-) = [x_+ - x]^+[x_- - y]^+$ and $Z_m = \eta_+ + \eta_- + \sum_{k=1}^m E_k$

$$\left|f(X_{n+1})H^{+,k_+}_{n+1,n+1}(H^{-,k_-}_{n+1,n+1}(\Phi)) - f(X_n)H^{+,k_+}_{n,n+1}(H^{-,k_-}_{n,n+1}(\Phi))\right|^p$$

$$= \left|\frac{f(X_{n+1})}{X^{k_+}_{+,n+1}X^{k_-}_{-,n+1}} - \frac{f(X_n)}{X^{k_+}_{+,n}X^{k_-}_{-,n}}\right|^p \left|H^{+,k_+}_{n,n+1}(H^{-,k_-}_{n,n+1}(\Phi))X^{k_+}_{+,n}X^{k_-}_{-,n}\right|^p$$

$$\leq \left|\frac{f(X_{n+1})}{X^{k_+}_{+,n+1}X^{k_-}_{-,n+1}} - \frac{f(X_n)}{X^{k_+}_{+,n}X^{k_-}_{-,n}}\right|^p p^{\phi}_{k_+,k_-}(Z_{n+1}, n+1)^p.$$

Here p is a polynomial of order $k_+ + k_-$. Now, recall

$$0 \leq X_{\pm,n} - X_{\pm,n-1} \leq \ell_n^{1/\alpha}[S_n]^+ + (a_n - a_{n-1})\eta_+^{1-1/\alpha}$$

How the moments are used: At infinity, there is an exchange of space variable and random variable. At zero, something more complicated (but similar) happens. For example,

$$\mathbf{1}_{X_+ > x} \leq \frac{X_+^p}{x^p}.$$

Simple interpolation inequalities for moment estimates

Moreover, we frequently apply the following basic interpolating inequalities: $\mathbf{1}_{\{y>x\}} \leq y^v x^{-v}$ for all $v \geq 0$ where we interpret the upper bound as 1 if $v = 0$. Also, if $y, z \geq 0$ then for all $r \in [0, 1]$

$$y \wedge z \leq y^r z^{1-r}$$
$$y \vee z \geq y^r z^{1-r}.$$

Define $(m_{\pm,n}, M_{\pm,n}) := (X_{\pm,n} \wedge X_{\pm,n+1}, X_{\pm,n} \vee X_{\pm,n+1})$ then $m_{\pm,n} = X_{\pm,n+1} \wedge X_{\pm,n} \geq \kappa X_{\pm,n}$ since $X_{\pm,n+1} \geq \kappa X_{\pm,n}$. Similarly, $M_{\pm,n} \leq \kappa^{-1} X_{\pm,n+1}$.

Some main points

- We have used an approximation method which converges exponentially fast to the limit of interest.
- In this construction which involves three different types of random variables, we recognize that the exponential random variables generate the supremum of the process.
- A finite dimensional calculus is obtained using these exponential random variables. The estimates explode polynomically fast.
- Using the Multi-level method as an interpolation technique we can use the exponential convergence in order to “kill ” the polynomial blow-up. (First interpolation)
- Two different situations: positive moments and negative moments are needed. In order to do this optimally additional simple algebraic interpolation inequalities are used.

V. Bally and L. Caramellino. Stochastic Integration by Parts.
Advanced Courses in Mathematics - CRM Barcelona. Springer International Publishing, 2016.

N. Bouleau and L. Denis. Energy image density property and the lent particle method for poisson measures.
Journal of Functional Analysis, 257(4):1144 – 1174, 2009.

L. Chaumont and J. Malecki. Entrance law of the excursion measure of the reflected process for some classes of Lévy processes.
Acta Appl Math, 169:59–77, 2020.

R. A. Doney and M. S. Savov. The asymptotic behavior of densities related to the supremum of a stable process.
Ann. Probab., 38(1):316–326, 2010.

N. Fournier and J. Printems. Absolute continuity for some one-dimensional processes.
Bernoulli, 16(2):343–360, 05 2010.

M. B. Giles. Multilevel Monte Carlo path simulation.
Oper. Res., 56(3):607–617, 2008.

J. I. González Cázares, A. Mijatović, and G. Uribe Bravo. Exact simulation of the extrema of stable processes.
Adv. in Appl. Probab., 51(4):967–993, 12 2019.

A. Kuznetsov. On extrema of stable processes.
Ann. Probab., 39(3):1027–1060, 2011.

A. Kuznetsov. On the density of the supremum of a stable process.
Stochastic Processes and their Applications, 123(3):986 – 1003, 2013.

D. Nualart. The Malliavin calculus and related topics. Probability and its Applications.
Springer-Verlag, New York, 2006.

D. Nualart and E. Nualart. Introduction to Malliavin Calculus, page 158–181.
Institute of Mathematical Statistics Textbooks. Cambridge University Press, 2018.

J. Pitman and G. Uribe Bravo. The convex minorant of a Lévy process.
Ann. Probab., 40(4):1636–1674, 2012.

R. Weron. On the Chambers-Mallows-Stuck method for simulating skewed stable random variables.
Statistics & Probability Letters, 28(2):165 – 171, 1996.