Second order McKean-Vlasov SDEs and kinetic Fokker-Planck-Kolmogorov equations

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- Let $(X_t)_{t\geqslant 0}$ be a continuous time-homogenous Markov process in $\mathbb{R}^d$ with generator $(\mathscr{L}, C_c^\infty)$ in the sense that for $f \in C_c^\infty$,

$$\mathbb{E}[f(X_{t+h}) - f(X_t)|\mathscr{F}_t] = h\mathscr{L}f(X_t) + o(h). \tag{1.1}$$

- One says $\mathscr{L}$ satisfies a maximum principle if for all $f \in C_c^\infty$ reaching a maximum at point $x_0 \in \mathbb{R}^d$, then $\mathscr{L}f(x_0) = 0$. By Courrège's theorem, $\mathscr{L}$ satisfies the maximum principle if and only if

$$\mathscr{L}f(x) = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\partial_{ij}^2 f(x) + \sum_{i=1}^d b_i(x)\partial_i f(x), \tag{1.2}$$

where a, b are measurable functions.

- Let $P_t(x,\cdot)$ be the transition probability of X_t starting from x at initial time 0. By (1.1), one sees that

$$\partial_t P_t f(x) = \lim_{h\downarrow 0}[P_{t+h}f(x) - P_t f(x)]/h = P_t \mathscr{L} f(x).$$

equivalently, $P_t(x,\cdot)$ solves the Fokker-Planck equation:

$$\partial_t P_t(x,\cdot) = \mathscr{L}^* P_t(x,\cdot), \quad \lim_{t\downarrow 0} P_t(x,\cdot) = \delta_{\{x\}}(\cdot). \tag{1.3}$$

- Kolmogorov(1931) and Feller(1936) constructed diffusion processes by solving (1.3) (now also called Fokker-Planck-Kolmogorov equation). (Analytic method! Smooth coefficients a, b.)

- Kolmogorov problem: Find a probabilistic construction of Markov process with given generator $\mathscr{L}$.

- Lévy and Itô's idea: Directly construct the sample path X_t by solving stochastic differential equations:

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sqrt{a}(X_t)\mathrm{d}W_t,$$

where W is a d-dimensional standard Brownian motion.

- Itô's integral equation(1942):

$$X_t = x + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sqrt{a}(X_s)\mathrm{d}W_s,$$

where b and a are Lipschitz continuous.

- SDE driven by general semimartingales (Doob, Watanabe...).
- Bernstein's stochastic difference equation(1934), Gihman(1947)....

- Stroock-Varadhan's martingale method(1969): Find a probability measure $\mathbb{P}$ over the sample path space $\mathbb{C}$ (the space of continuous functions) so that for each test function f:

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \mathscr{L} f(X_s)\mathrm{d}s$$

  is a martingale under $\mathbb{P}$ with respect to the natural filtration.
- Advantage: Avoid the use of stochastic integral, more flexible.
- Martingale solutions = Weak solution of SDEs.
- Hille-Yosida semigroup and Dirichlet theory: Fukushima(1979), Ma, Röckner(1990)...

- We continue to consider SDE:

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sqrt{a}(X_t)\mathrm{d}W_t,$$

- Strong well-posedness: b and σ Lipschitz, singular drift b and Sobolev diffusion σ (Krylov and Röckner, Z., ...)
- Weak well-posednewss (Stroock-Varadhan):

  bounded b + non-degenerate continuous σ.

- Weak existence (Krylov):

  bounded b + non-degenerate measurable σ.

- Through Markov selection to find a Markov process.

- Classical Newtonian mechanics equation

$$\underbrace{F = ma}_{\text{Physics}} \Longleftrightarrow \underbrace{\ddot{X}_t = F(X_t, \dot{X}_t)}_{\text{Mathematics}},$$

where F stands for the external force, m for the mass and a for the acceleration, X_t stands for the position of particles at time t.

- Second order ODE with stochastic Brownian force:

$$\ddot{X}_t = F(X_t, \dot{X}_t) + \sigma(X_t, V_t)\dot{W}_t.$$

- Langevin equation (the first SDE):

$$\ddot{X}_t = -\gamma \dot{X}_t + \eta \dot{W}_t.$$

- First order SDE (degenerate):

$$\dot{X}_t = V_t, \ \dot{V}_t = F(X_t, V_t) + \sigma(X_t, V_t)\mathrm{d}W_t. \tag{1.4}$$

- The law ρ_t of (X_t, V_t) satisfies the Fokker-Planck equation

$$\partial_t \rho + v \cdot \nabla_x \rho = \partial_{v_i}\partial_{v_j}(a^{ij}\rho) + \mathrm{div}_v(F\rho),$$

  where $a^{ij} = \sigma^{ik}\sigma^{jk}/2$.
- Important observation: If $\sigma(x, v) = \sigma(x)$, then

$$\partial_t \rho + v \cdot \nabla_x \rho = a^{ij}(x)\partial_{v_i}\partial_{v_j}\rho + \mathrm{div}_v(F\rho),$$

- Aim: Establish the weak well-posedness of (1.4) with discontinuous F and σ through De-Giorgi's theory of kinetic FPKEs.

Second order McKean-Vlasov SDEs

- Consider the following second order DDSDE:

$$\dot{X}_t = V_t, \ \mathrm{d}V_t = b_Z(t, Z_t)\mathrm{d}t + \sqrt{2a_Z}(t, X_t)\mathrm{d}W_t, \tag{2.1}$$

where $Z = (X, \dot{X})$ and

$$b_Z(t, z) := \int_{\mathbb{R}^{2d}} b(t, z, \rho_{Z_t}(z), z')\mu_{Z_t}(\mathrm{d}z'), \tag{2.2}$$

and

$$a_Z(t, x) := \int_{\mathbb{R}^d} a(t, x, \rho_{X_t}(x), z')\mu_{Z_t}(\mathrm{d}z'), \tag{2.3}$$

and

$$\mu_{Z_t}(\mathrm{d}z) = \mathbf{P} \circ Z_t^{-1}(\mathrm{d}z) = \rho_{Z_t}(z)\mathrm{d}z.$$

- We call SDE (2.1) density-distribution dependent SDEs.

- By Itô's formula, the density $\rho(t,z) := \rho_{Z_t}(z)$ solves the following nonlinear kinetic Fokker-Planck-Kolmogorov equation (FPKE):

$$\partial_t \rho = \partial_{v_i}\partial_{v_j}(\bar{a}_{ij}(\rho)\rho) - v \cdot \nabla_x \rho + \mathrm{div}_v(\bar{b}(\rho)\rho), \tag{2.4}$$

  where for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$,

$$\bar{a}(\rho; t, x) := \int_{\mathbb{R}^{2d}} a(t, x, \langle\rho\rangle(t,x), z')\rho(t,z')(\mathrm{d}z'),$$

  and for $(t,z) \in \mathbb{R}_+ \times \mathbb{R}^{2d}$,

$$\bar{b}(\rho; t, z) := \int_{\mathbb{R}^{2d}} b(t, z, \rho(t,z), z')\rho(t,z')(\mathrm{d}z').$$

- Here $\langle\rho\rangle(t,x) := \int_{\mathbb{R}^d} \rho(t,x,v)\mathrm{d}v$ stands for the mass density.

- Distribution-DSDE has been studied extensively (Mishura, Veretennikov, Li, Wang, Huang, Bao, ...)
- Singular kinetic distribution-SDE (Hao-Z.-Zhu-Zhu(2021)).
- Density-DSDE was firstly studied by Barbu and Röckner(2018) for providing a probabilistic representation for nonlinear FPKEs. (Based on superposition principle).
- In the scope of PDEs, the kinetic FPKEs have been studied extensively by many peoples (Golse, Imbert, Mouhot, Lanconelli, Pascucci, Polidoro, Silvestre, Vasseur, Zhang, $\cdots$).

Definition 1 (Weak solutions)

Let $\nu \in \mathcal{P}(\mathbb{R}^{2d})$ and $\mathfrak{F} := (\Omega, \mathscr{F}, \mathbf{P}; (\mathscr{F}_t)_{t\geqslant 0})$ be a stochastic basis, $Z_t = (X_t, V_t)$ and W_t be $\mathbb{R}^{2d}$ and $\mathbb{R}^d$-valued $\mathscr{F}_t$-adapted processes, respectively. We call $(\mathfrak{F}, Z, W)$ a weak solution of DDSDE (2.1) with initial distribution ν if

(i) $\mathbf{P} \circ Z_0^{-1} = \nu$, and for Lebesgue almost all $t \geqslant 0$,

$$\mathbf{P} \circ Z_t^{-1}(\mathrm{d}z) = \rho_{Z_t}(z)\mathrm{d}z, \quad \mathbf{P} \circ X_t^{-1}(\mathrm{d}x) = \rho_{X_t}(x)\mathrm{d}x.$$

(ii) W is a standard d-dimensional $\mathscr{F}_t$-Brownian motion.

(iii) For all $t \geqslant 0$, it holds that $X_t = X_0 + \int_0^t V_s \mathrm{d}s$ and

$$V_t = V_0 + \int_0^t b_Z(s, Z_s)\mathrm{d}s + \int_0^t \sigma_Z(s, X_s)\mathrm{d}W_s, \ \mathbf{P} - a.s.,$$

where b_Z and σ_Z are defined by (2.2) and (2.3), respectively.

- Fix $N \in \mathbb{N}$. For multi-index $\boldsymbol{p} = (p_1, \cdots, p_N) \in (0, \infty]^N$, we define

$$\|f\|_{\mathbb{L}^{\boldsymbol{p}}} := \left[\int_{\mathbb{R}} \left[\int_{\mathbb{R}} \cdots \left[\int_{\mathbb{R}} |f(z_1, \cdots, z_N)|^{p_N} \mathrm{d}z_N \right]^{\frac{p_{N-1}}{p_N}} \cdots \mathrm{d}z_2 \right]^{\frac{p_1}{p_2}} \mathrm{d}z_1 \right]^{\frac{1}{p_1}}.$$

- Note that for any permutation $\boldsymbol{p}'$ of $\boldsymbol{p}$,

$$\|f\|_{\mathbb{L}^{\boldsymbol{p}}} \neq \|f\|_{\mathbb{L}^{\boldsymbol{p}'}}.$$

- For any multi-indices $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r} \in (0, \infty]^N$ with $\frac{1}{\boldsymbol{p}} + \frac{1}{\boldsymbol{r}} = \frac{1}{\boldsymbol{q}}$, the following Hölder's inequality holds

$$\|fg\|_{\mathbb{L}^{\boldsymbol{q}}} \leqslant \|f\|_{\mathbb{L}^{\boldsymbol{p}}} \|g\|_{\mathbb{L}^{\boldsymbol{r}}}.$$

- For any multi-indices $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r} \in [1, \infty]^N$ with $\frac{1}{\boldsymbol{p}} + \frac{1}{\boldsymbol{r}} = \boldsymbol{1} + \frac{1}{\boldsymbol{q}}$, the following Young's inequality holds

$$\|f * g\|_{\mathbb{L}^{\boldsymbol{q}}} \leqslant \|f\|_{\mathbb{L}^{\boldsymbol{p}}} \|g\|_{\mathbb{L}^{\boldsymbol{r}}}.$$

(H$_1$) For any $m \in \mathbb{N}$ and bounded domain $Q \subset \mathbb{R}_+ \times \mathbb{R}^{2d} \times \mathbb{R}^{2d}$, it holds

$$\lim_{h\to 0} \Big\| \sup_{r,r'\leqslant m, |r-r'|\leqslant h} |b(\cdot,\cdot,r,\cdot) - b(\cdot,\cdot,r',\cdot)| \Big\|_{L^1(Q)} = 0,$$

and for all $(t,z,r,z') \in \mathbb{R}_+ \times \mathbb{R}^{2d} \times \mathbb{R}_+ \times \mathbb{R}^{2d}$,

$$|b(t,z,r,z')| \leqslant h(t, z - z') \text{ with } \|h\|_{\mathbb{L}^{q_1}_t(\mathbb{L}^{\boldsymbol{p}_1}_z)} \leqslant \kappa_2,$$

where $q_1 \in (2,4)$ and $\boldsymbol{p}_1 \in (2,\infty)^{2d}$ satisfy $\boldsymbol{a} \cdot \frac{1}{\boldsymbol{p}_1} + \frac{2}{q_1} < 1$. Here $\boldsymbol{a} = (3,\cdots,3,1,\cdots,1) \in \mathbb{R}^{2d}$.

(H$_2$) For each $(t,x,z) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^{2d}$, $[0,\infty) \ni r \mapsto a(t,x,r,z) \in \mathbb{M}^d_{\rm sym}$ is continuous and

$$\kappa_0|\xi|^2 \leqslant \xi \cdot a(t,x,r,z')\xi \leqslant \kappa_1|\xi|^2, \quad \xi \in \mathbb{R}^d.$$

Theorem 2 (Existence of weak solutions)

Under **(H$_1$)** *and* **(H$_2$)**, *for any* $\nu \in \mathcal{P}(\mathbb{R}^{2d})$, *there exists at least one weak solution to SDE* (2.1) *with initial distribution* ν *in the sense of Definition 1. Moreover, the density* ρ *enjoys the following regularity: for any* $\alpha \in (0,1)$ *and* $(q, \boldsymbol{p}) \in (1,\infty)^{1+2d}$ *satisfying*

$$\frac{2}{q} < 1 + \alpha, \quad \frac{2}{q} + \boldsymbol{a} \cdot (\frac{1}{\boldsymbol{p}} - \mathbf{1}) > 2\alpha,$$

it holds that

$$\|\rho \mathbf{1}_{[0,T]}\|_{\mathbb{L}^q_t(\mathbf{B}^{\alpha}_{\boldsymbol{p};\boldsymbol{a}})} < \infty, \ T > 0, \tag{2.5}$$

where $\mathbf{B}^{\alpha}_{\boldsymbol{p};\boldsymbol{a}}$ *is the anisotropic Besov space.*

Remark 1: Chaudru de Raynal and Menozzi(2018): Weak well-posedness for Hölder diffusions and singular drifts $b \in L^q_t(L^p_{x,v})$ with $\frac{2}{q} + \frac{4d}{p} < 1$.

Remark 2: Ling and Xie(2020) firstly used the mixed $L^{\boldsymbol{p}}$-norm for the study of SDE. It is useful for singular interaction particle systems.

Kinetic Fokker-Planck-Kolmogorov equations

- Consider the following linear kinetic FPKE of divergence form:

$$\partial_t u = \mathrm{div}_v(a \cdot \nabla_v u) + v \cdot \nabla_x u + b \cdot \nabla_v u + f, \tag{3.1}$$

where

$$a : \mathbb{R}^{1+2d} \to \mathbb{M}^d_{\mathrm{sym}}, \ b : \mathbb{R}^{1+2d} \to \mathbb{R}^d, \ f : \mathbb{R}^{1+2d} \to \mathbb{R}$$

are Borel measurable functions and satisfies

$$\kappa_0 \mathbb{I} \leqslant a \leqslant \kappa_1 \mathbb{I}, \tag{3.2}$$

and for any bounded $Q \subset \mathbb{R}^{1+2d}$,

$$b\mathbf{1}_Q \in \mathbb{L}^2, \ \ f\mathbf{1}_Q \in \mathbb{L}^1.$$

- For an open set $Q \subset \mathbb{R}^{1+2d}$, we introduce local energy space

$$\mathscr{V}_Q := \left\{ f \in \mathbb{L}^1_{loc} : \|f\|_{\mathscr{V}_Q} := \|\mathbf{1}_Q f\|_{\mathbb{L}^\infty_t(\mathbb{L}^2_z)} + \|\mathbf{1}_Q \nabla_v f\|_{\mathbb{L}^2} < \infty \right\}.$$

- For simplicity, we write for any $T > 0$,

$$\mathscr{V}_T := \mathscr{V}_{[0,T]\times\mathbb{R}^{2d}}, \quad \mathscr{V} := \mathscr{V}_{\mathbb{R}^{1+2d}}, \quad \mathscr{V}_{loc} := \cap_{\text{bounded } Q} \mathscr{V}_Q,$$

Definition 3 (Weak solutions of FPKE)

A function $u \in \mathscr{V}_{loc} \cap \mathbb{L}^\infty_{loc}$ is called a weak solution of PDE (3.1) if for any $\varphi \in C^\infty_c(\mathbb{R}^{1+2d})$,

$$\begin{aligned} -\int u \partial_t \varphi = &- \int \langle a \cdot \nabla_v u, \nabla_v \varphi \rangle - \int u(v \cdot \nabla_x \varphi) \\ &+ \int (b \cdot \nabla_v u)\varphi + \int f\varphi. \end{aligned} \tag{3.3}$$

Theorem 4 (L^∞-estimates for real function f)

Suppose that (3.2) *and* $f \in \mathbb{L}_t^{q_0}(\mathbb{L}_z^{\boldsymbol{p}_0})$ *and* $\|b\|_{\mathbb{L}_t^{q_1}(\mathbb{L}_z^{\boldsymbol{p}_1})} \leqslant \kappa_1$*, where*

$$\boldsymbol{a} \cdot \frac{1}{\boldsymbol{p}_0} + \frac{2}{q_0} < 2, \quad \boldsymbol{a} \cdot \frac{1}{\boldsymbol{p}_1} + \frac{2}{q_1} < 1.$$

For any $T > 0$*, there is a unique weak solution* u *to PDE* (3.1) *with initial value* $u(t)|_{t\leqslant 0} = 0$ *such that for some* $C = C(d, T, \kappa_0, \kappa_1, q_i, \boldsymbol{p}_i) > 0$*,*

$$\|u\|_{\mathbb{L}_T^\infty} + \|u\|_{\mathscr{V}_T} \lesssim_C \|f\|_{\mathbb{L}_T^{q_0}(\mathbb{L}_z^{\boldsymbol{p}_0})}.$$

Remark: Energy estimate for $(u - \kappa)^+$ (weak sub-solution) and comparison principle, Golse-Imbert-Mouhot-Vasseur(2019): Harnack estimate for kinetic equations with bounded b, f.

Theorem 5 (L^∞-estimates for distribution-valued f)

Suppose that (3.2) *and* $\|b\|_{\mathbb{L}_t^{q_1}(\mathbb{L}_z^{\boldsymbol{p}_1})} \leqslant \kappa_1$ *for some* $\boldsymbol{a} \cdot \frac{1}{\boldsymbol{p}_1} + \frac{2}{q_1} < 1$ *and*

$$f \in \mathbb{L}_t^{q_0}(\mathbf{B}^{-\alpha_0}_{\boldsymbol{p}_0;\boldsymbol{a}}),$$

where $\alpha_0 \in (0,1)$ *and*

$$1 - \alpha_0 < \frac{2}{q_0}, \ \frac{2}{q_0} + \boldsymbol{a} \cdot \frac{1}{\boldsymbol{p}_0} < 2 - 2\alpha_0.$$

For any $T > 0$*, there is a unique weak solution u to PDE* (3.1) *with initial value* $u(t)|_{t\leqslant 0} = 0$ *such that for some* $C = C(d, T, \kappa_0, \alpha_0, \kappa_1, q_i, \boldsymbol{p}_i) > 0$*,*

$$\|u\|_{\mathbb{L}^\infty_T} + \|u\|_{\mathscr{V}_T} \lesssim_C \|f\|_{\mathbb{L}_T^{q_0}(\mathbf{B}^{-\alpha_0}_{\boldsymbol{p}_0;\boldsymbol{a}})}.$$

Key point: Energy estimates for $((u-\kappa)^+)^2$ (weak solution).

Theorem 6 (Stability)

Let (a_n, b_n, f_n) be a sequence of coefficients that satisfy the following assumptions:

(i) *a_n satisfies (3.2) uniformly in n and b_n, f_n are uniformly bounded.*

(ii) *(a_n, b_n, f_n) converges to (a, b, f) in Lebesgue measure as $n \to \infty$.*

Let u_n and u be the respective weak solutions of PDE (3.1) corresponding to (a_n, b_n, f_n) and (a, b, f) with zero initial value. Then for any bounded domain $Q \subset \mathbb{R}_+ \times \mathbb{R}^{2d}$,

$$\lim_{n\to\infty} \sup_{(t,z)\in Q} |u_n(t,z) - u(t,z)| = 0.$$

Generalized martingale problem

- Consider the following backward Kolmogorov equation:

$$\partial_t u + \mathrm{tr}(a \cdot \nabla_v^2 u) + v \cdot \nabla_x u + b_\rho \cdot \nabla_v u = f, \quad u(T) = 0, \qquad (4.1)$$

where for a family of density functions $\rho_t(z)$ in $\mathbb{R}^{2d}$,

$$b_\rho(t, z) := \int_{\mathbb{R}^{2d}} b(t, z, \rho_t(z), z')\rho_t(z')\mathrm{d}z'.$$

- By Theorem 6, there is a unique solution

$$u_\rho^f \in \widetilde{\mathscr{V}}_T \cap C_b([0, T] \times \mathbb{R}^{2d}).$$

Here the continuity of u_ρ^f is crux for generalized martingale problem.

Definition 7 (Generalized martingale problem)

Let $s \geqslant 0$ and $\nu \in \mathcal{P}(\mathbb{R}^{2d})$. A probability measure $\mathbb{P} \in \mathcal{P}(\mathbb{C})$ is called a generalized martingale solution of DDSDE (2.1) starting from (s, ν) if

(i) $\mathbb{P} \circ \omega_s^{-1} = \nu$, and for Lebesgue almost all $t \geqslant s$,

$$\mathbb{P} \circ \omega_t^{-1}(\mathrm{d}z) = \rho_t(z)\mathrm{d}z.$$

(ii) For any $T > s$ and $f \in C_c^\infty(\mathbb{R}^{2d})$, the process

$$M_t := u_\rho^f(t, \omega_t) - u_\rho^f(s, \omega_s) - \int_s^t f(\omega_r)\mathrm{d}r, \quad t \in [s, T],$$

is a $\mathcal{B}_t$-martingale with respect to $\mathbb{P}$.

The set of all the generalized martingale solutions $\mathbb{P} \in \mathcal{P}(\mathbb{C})$ with initial distribution ν at time s is denoted by $\widetilde{\mathcal{M}}_{s,\nu}^{a,b}$.

(H_3) $a = a(t,x)$ is independent of (r,z') and uniformly elliptic, and b is bounded measurable and satisfies that for some $C > 0$ and all $(t,z,z') \in \mathbb{R}_+ \times \mathbb{R}^{2d} \times \mathbb{R}^{2d}$ and $r, r' \geqslant 0$,

$$|b(t,z,r,z') - b(t,z,r',z')| \leqslant C|r - r'|.$$

Theorem 8 (Well-posedness of generalized martingale problems)

Under **(H_3)**, *for any $s \geqslant 0$ and $\nu(\mathrm{d}z) = \rho_0(z)\mathrm{d}z$, where $\rho_0 \in C_b^1(\mathbb{R}^{2d})$, there is a unique generalized martingale solution $\mathbb{P} \in \widetilde{\mathcal{M}}_{s,\nu}^{a,b}$ in the sense of Definition 7. Moreover, its density $\rho_t(z)$ enjoys the regularity (2.5) and solves the following nonlinear kinetic FPKE in the distributional sense:*

$$\partial_t \rho_t = \partial_{v_i}\partial_{v_j}(a_{ij}\rho_t) - v \cdot \nabla_x \rho_t + \mathrm{div}_v(b_\rho \rho_t), \ \ t \geqslant s,$$

where $b_\rho(t,z) := \int_{\mathbb{R}^{2d}} b(t,z,\rho_t(z),z')\rho_t(z')\mathrm{d}z'$.

- If b does not depend on the density variable r, then we can drop the regularity assumption on the initial distribution ν.
- Suppose that $a = a(x)$ is uniformly elliptic and $b = b(z)$ is bounded measurable, then for each $z \in \mathbb{R}^{2d}$, the following second order SDE admits a unique generalized martingale solution $\mathbb{P}_z \in \widetilde{\mathcal{M}}^{a,b}_{0,\delta_z}$:

$$\mathrm{d}\dot{X}_t = b(Z_t)\mathrm{d}t + \sqrt{a}(X_t)\mathrm{d}W_t, \quad Z_0 = z.$$

- Let b_ε and a_ε be the smooth approximation of b and a. Consider

$$\mathrm{d}\dot{X}^\varepsilon_t = b_\varepsilon(Z^\varepsilon_t)\mathrm{d}t + \sqrt{a_\varepsilon}(X^\varepsilon_t)\mathrm{d}W_t, \quad Z^\varepsilon_0 = z.$$

  By the uniqueness, for each $z \in \mathbb{R}^{2d}$, the law of Z^ε weakly converges to $\mathbb{P}_z$ (not necessarily subtracting a subsequence). In particular, $(\mathbb{P}_z)_{z\in\mathbb{R}^{2d}}$ forms a family of strong Markov processes.

Thank you for your attention!